On some mathematical modelling of self-field MPD thrusters

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Dedicated to the memory of Jindrič Nečas

We are concerned with the mathematical modelling of high-enthalpy compressible plasma flows of magnetoplasmadynamic (MPD) rocket thrusters. A closed system of partial differential equations is associated with corresponding two-fluid flows which are subjected to strong electromagnetic fields. The numerical solution of the full original model requires nowadays huge computational costs. Therefore we use a heterogeneous domain decomposition where the corresponding coupled model is based on simplified conservation laws in a large subdomain. We establish new transmission conditions for the coupled MPD model, which appear as a byproduct of a detailed asymptotic analysis of the artificial interfaces that arise by domain decomposition. Moreover, a complete coupled mathematical MPD model is formulated, including appropriate outer boundary conditions. The well-posedness of this model is discussed by using mathematical as well as physical arguments.

1 Introduction

In this work, we are concerned with domain decomposition applied to the mathematical modelling of compressible magnetoplasmadynamic (MPD) flows. Specifically, we are interested in modelling flows of MPD rocket thrusters or accelerators. The geometry and the principle of the functionality of the device are described in Sect. 2. Beyond the motion of the heavy particles (atoms and ions of the propelling gas), we have to include an electron energy field. Consequently, the corresponding flow is a two-phase fluid flow, since the different components of the plasma behave differently. In addition, the influence of a strong magnetic field must be taken into account. The general mathematical model for such an MPD flow is described in Sect. 3. Since the plasma occupies a rotationally symmetric domain, we use cylindrical coordinates, and, assuming that the flow itself is also rotationally symmetric, we reduce the 3D model to a 2D one. This is still rather complex and includes twelve partial differential equations in the form of conservation laws, and four algebraic state equations, as described in Sect. 3. Turbulence effects are not taken into consideration. This system of partial differential equations is closed due to the state equations, and together with initial and specific boundary conditions, it represents a nonlinear mathematical model for our two-phase fluid flow which is subjected to a strong electrical Lorentz force field. The huge computational costs for the numerical simulation of such a complete model can be avoided by using heterogeneous domain decomposition methods. The \((r, z)\)-flow field \(\Omega\) is decomposed into two disjoint subdomains: A relatively small domain \(\Omega_1\) corresponding to the interior of the proper MPD thruster, called the near field, where the full PDE system is used, and the complementary subdomain \(\Omega_2\), called the far field, corresponding to the test tank, where a simplified system of equations is employed. The latter is obtained by making use of the physical assumption that the magnetic field as well as the viscosity effects for the heavy-particle component of the plasma are both negligible small in \(\Omega_2\). In this way, we are led to a coupled problem which is more feasible for the numerical simulation.

The solution procedure of the coupled problem consists of an iterative alternating solution of the subproblems in the nonoverlapping regions. Since the subproblems are modelled by extensions of the Navier-Stokes and the Euler equations, respectively, one needs transmission conditions that must serve as respective boundary conditions, describing the interaction between the different model zones with viscous and inviscid flow fields in \(\Omega_1\) and \(\Omega_2\).

For this purpose, we employ a new approach, based on singular perturbation theory. Since the MPD problem is extremely complex, we start our investigation with simpler models with conservation laws, in order to get the main ingredients for the original MPD-model.

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In Sect. 4 we investigate some Cartesian 2D coupled models associated with one-phase compressible flows. First, we discuss in Sects. 4.1–4.2 the well-posedness of different initial boundary value problems associated with 2D compressible flows described by the Navier-Stokes and by the Euler equations. This topic has a long history of more than two centuries [32] and is still not completed. However, physical arguments may be used when not enough mathematical information is available. In Sect. 4.3 we establish transmission conditions for 2D coupled compressible flows by using our approach that is based on singular perturbation analysis. This is in agreement with physical principles of compressible flows. In particular, the Mach number is a key parameter involved in our analysis. We then discuss the possible extensions of our investigations to the original MPD model concerning interfaces and well-posedness.

Sect. 5 is dedicated to the formulation of a well-posed coupled model for the MPD flow. A detailed analysis of the corresponding artificial interface is performed in order to establish correct transmission conditions. Our equations expressing the flux conservation across the interface are new. They are derived by equating all the possible normal fluxes corresponding to the left and right sides of the coupling boundary. Possible continuity conditions for the flow variables are incorporated into initial conditions for appropriate transition layer corrections. Finally, we formulate a full, coupled MPD model, including specific outer boundary conditions. In order to do this we make use of the foregoing discussions concerning well-posedness.

In Sect. 6 we present an approximate Navier-Stokes/Euler solution obtained by heterogeneous domain decomposition. We investigate the behavior of the inviscid/viscous coupled solution at the interface, in order to get qualitative information on the quality of the coupling, which motivates the theoretical investigation and the implementation of the transition layer correction method for the a-posteriori improvement of the coupled solution.

This work contains a substantial original part, in particular in Sects. 5 and 6. At the same time, it is a survey on the topic, including also known literature. In many respects this work depends on further contributions of the authors [3, 6–18, 26].

## 2 Magnetoplasmadynamic flows. The MPD thruster

A magnetoplasma is an ionized gas in a strong magnetic field. Classical examples of natural magnetoplasmas are the sun and the interplanetary solar wind. For scientific and technical purposes, magnetoplasmas have also been created artificially. The treatment of Magnetoplasma Dynamics (MPD) requires knowledge in fluid dynamics, particle dynamics, thermodynamics, and electromagnetism. Clearly, the MPD flows are extremely complex phenomena and their understanding is far from being complete. Even the modelling of the MPD flows is currently not sufficiently clear. Nevertheless, a huge amount of work has been dedicated to this important field (see, e.g., [37] and the references therein), and many practical applications are to be investigated. Among these applications, we mention the controlled fusion and the MPD thrusters. Here we are mainly concerned with MPD thrusters which are candidates for propelling manned spaceships to Mars.

Laboratory experiments with MPD thrusters have been developed since the 1970’s at the Institute of Space Systems at the University of Stuttgart. For interplanetary space missions, high thrust levels are required for the thruster systems of rockets. MPD thrusters are considered for such missions due to their simplicity and thrust efficiency. In order to increase the thrust efficiency, much effort is still needed to achieve a better understanding of the MPD thrusters. Specifically, this can be done by further experimental work, by improving the mathematical model associated with compressible flows inside the MPD accelerators, as well as by analytical and numerical results for the simulation of MPD flows. We report here some corresponding achievements.

The MPD thruster (see Fig. 1 below) is an axisymmetric device that exploits an electrical arc discharge between two electrodes (anode and cathode) to heat up a gas to a temperature of several ten-thousand Kelvin.

The resulting plasma carries an electric current of density $j$ which induces an azimuthal magnetic field $B$. Thus, the plasma is accelerated by its thermal expansion as well as by the electromagnetic Lorentz force field $j \times B$, and so a very high exhaust velocity is achieved, necessary for spacecraft propelling. Argon is typically used as propellant for MPD thrusters because of its low specific ionization energy level. In laboratory experiments, the plasma is accelerated into a test tank so that the flow domain is bounded; see Fig. 2 for an axial section of the geometry of the MPD thruster.

Due to the symmetry we have represented only the upper part of the section. The cold gas (argon) is injected into the device, and the resulting plasma is extracted from the tank outlet by a pump situated at the right end of the tank. Here, we have an example of a two phase fluid flow, since the heavy particles (atoms and ions) and the electrons behave differently. In addition, we have to take into account the influence of the electromagnetic field. For a more complete description of the MPD thruster we refer the reader to [26].

## 3 Mathematical modelling of the MPD flow

In this section we present a mathematical model associated with the three-dimensional (3D) flow inside an MPD thruster, extended into a test tank, as described above. Since the fluid occupies a rotationally symmetric domain, we assume that the MPD flow is also rotationally symmetric and we use cylindrical coordinates to reduce the model to a two-dimensional (2D) one. Nevertheless, our mathematical 2D model is still very complex. It contains several partial differential equations (conservation laws) and, moreover, algebraic state equations, all together describing the MPD flow in the interior of the 2D domain (see Sect. 3.1 below). In addition, initial conditions and specific boundary conditions must be added to complete the mathematical...
model. Note that for the time being there is no existence result for such a complex model. We shall discuss this matter later on.

Even for simpler Navier-Stokes models associated with a compressible one-phase fluid flow, without any magnetic field, there is no complete existence theory available yet either (see [21,32]). However, we shall assume that every model we consider has a unique and sufficiently smooth solution whenever all the data are sufficiently smooth. Moreover, we shall introduce coupled (transmission) problems by coupling the complete model with some simplified models, in order to reduce the computational cost of corresponding numerical simulations. For such coupled problems some jumps of the solutions are expected to appear at the interfaces as will be seen in what follows.

For the presentation of the rotationally symmetric 2D model, we shall mainly use the doctoral thesis [26]. As indicated before, we have a two phase fluid flow, since the electron component behaves differently as compared to the heavy-particle component of the MPD fluid. Let us begin with the

3.1 Modelling of the heavy-particle flow

3.1.1 Conservation of mass

Denote by \( s_i \) \((i = 0, \ldots, 6)\) the species of heavy particles \((s_0: \text{The atom of neutral argon } \text{Ar}^0, \text{and } s_i: \text{The ions } \text{Ar}^{i+}, \text{for } i = 1, \ldots, 6)\). Let \( n_i \) denote the density of \( s_i \), depending on space and time. Correspondingly, let \( n_h = \sum_{i=0}^{6} n_i \) denote the heavy-particle density, and \( n_e \) the electron density. The functions \( n_i \) satisfy the equations

\[
\frac{\partial n_i}{\partial t} = - \text{div}(n_i \mathbf{v}) - \text{div} j_{D,i} + \omega_i \quad \text{for } i = 0, \ldots, 6,
\]

i.e., the temporal variation of \( n_i \) is due to convection, mass diffusion, and reactions. Here, \( \mathbf{v} \) represents the velocity of the heavy-particle component of the fluid. The functions \( j_{D,i} \) are defined by

\[
j_{D,i} = j'_{D,i} - \zeta_i \sum_{k=0}^{6} j'_{D,k} \quad \text{for } i = 0, \ldots, 6,
\]
where $\xi_i = \frac{n_i}{n_h}$, and the quantities $J_{D,i}$, defining the diffusion stream, are given by
\[
J_{D,i} = -n_h D_{im} \nabla \Psi_i
\]
with $\Psi_i := \frac{n_i}{n_h + n_e}$ and with the coefficients $D_{im}$ depending on $n_i$, $T_h$, and $T_e$. Here $T_h$, $T_e$ denote the temperatures of the heavy-particle component and the electron component of the fluid, respectively. Note that $J_{D,i}$ also depends on $n_i$, $T_h$, $T_e$.

Assuming that the plasma is quasineutral, we obtain the relation
\[
n_e = \sum_{i=1}^{6} n_i
\]
between the densities of the different species (note that $i$ represents the number of (positive) charges of $s_i$).

As a consequence, we get the relation
\[
\sum_{i=0}^{6} J_{D,i} = 0.
\] (3.3)
The term $\omega_i$ in eq. (3.1) represents the production of electrons and ions, resulting from collisions and recombinations.
\[
\begin{align*}
\omega_0 &= -n_0 n_e k_{f,1} + n_1 n_e^2 k_{b,1}, \\
\omega_i &= -n_i n_e k_{f,i+1} + n_{i+1} n_e^2 k_{b,i+1} - n_i n_e^2 k_{b,i} + n_{i-1} n_e k_{f,i} \quad (i = 1, \ldots, 5), \\
\omega_6 &= -n_0 n_e^2 k_{b,6} + n_5 n_e k_{f,6},
\end{align*}
\]
where $k_{f,i}$ and $k_{b,i}$ are the reaction rates for ionizations and recombinations (which lead to the neutralization of ions). The rates $k_{f,i}$ and $k_{b,i}$ depend on $T_e$. We also point out that $\omega_i$ depends on $n_i$, $T_e$ and, finally,
\[
\sum_{i=0}^{6} \omega_i = 0.
\] (3.4)

If we denote $\rho = m_h n_h$ with $m_h$ the heavy particle mass, then summing in (3.1) (see also (3.3) and (3.4)) we get the classical conservation of mass for the heavy-particle component of the fluid,
\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0.
\] (3.5)
The function $\rho$ is the mass density of the heavy fluid, i.e. the heavy-particle component of the compressible plasma flow.

### 3.1.2 Balance of momentum

The balance of momentum is expressed by the vector equation
\[
\frac{\partial}{\partial t} (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = \text{div} \mathbf{T} + \mathbf{j} \times \mathbf{B},
\] (3.6)
where $\mathbf{T}$ denotes the viscous stress tensor, defined by
\[
\mathbf{T} = \mu_h \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) + \lambda_h (\text{div} \mathbf{v}) \mathbf{I}
\] (3.7)
with $\mathbf{I}$ the identity and $\mu_h$, $\lambda_h$ the viscosity coefficients of the heavy-particle fluid, satisfying the thermodynamical condition
\[
3 \lambda_h + 2 \mu_h = 0.
\]

Herewith, we can express $\mathbf{T}$ in terms of $\mu_h$ (the dynamical viscosity) only, which depends on $n_i$, $T_h$, $T_e$. Note that (3.6) is the classical momentum conservation law (see, e.g., [20, 21, 32]), here with the total pressure $p = p_h + p_e$ consisting of the heavy-particle pressure $p_h$ and of the electron pressure $p_e$; and with the Lorentz force field $\mathbf{j} \times \mathbf{B}$. Recall that $\mathbf{j}$ is the electric current density, and $\mathbf{B}$ is the magnetic field. Since the fluid occupies a rotationally symmetric domain and we also assume rotational symmetry for the fields, we can use cylindrical coordinates to express (3.6) as a system of 2 scalar equations only. The Cartesian coordinates $(x, y, z)$ and the cylindrical coordinates $(r, \phi, z)$ of the same point $M$ are related by
\[
x := r \cos \phi, \quad y := r \sin \phi, \quad z := z.
\] (3.8)
In order to introduce some notation, we simply use the picture given in Fig. 3. An arbitrary vector \( \mathbf{v} \) will be expressed as
\[
\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z = \mathbf{v}_r \mathbf{e}_r + v_\phi \mathbf{e}_\phi + v_z \mathbf{e}_z.
\] (3.9)

Using the transformation of coordinates
\[
v_x = v_r \cos \phi - v_\phi \sin \phi,
\]
\[
v_y = v_r \sin \phi + v_\phi \cos \phi,
\]
\[
v_z = v_z,
\] (3.10)
we obtain for a vector field \( \mathbf{v} \) the relation
\[
\text{div} \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.
\] (3.11)

As already mentioned, based on the symmetry of the domain and of the data, we assume here the rotational symmetry of the flow. In particular, the heavy-particle velocity \( \mathbf{v} \) does not depend on \( \phi \), i.e., \( \mathbf{v} = v_r \mathbf{e}_r + v_z \mathbf{e}_z \), and therefore
\[
\text{div} \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z}.
\] (3.11)

Since the flow variables are assumed to be independent of \( \phi \), the 3D domain is replaced by the 2D domain \( \Omega \) in the \((r, z)\)-plane as shown in Fig. 4.

We recall that the magnetic field is now purely azimuthal, i.e.,
\[
\mathbf{B} = B \mathbf{e}_\phi,
\] (3.12)
with \( B \) depending on \((r, z, t)\). This, combined with the Maxwell equation
\[
\text{rot} \mathbf{B} = \mu_0 \mathbf{j},
\]
leads us to the following relations for the components of \( \mathbf{j} \times \mathbf{B} \):
\[(j \times B)_r = - \frac{B}{\mu_0 r} \frac{\partial}{\partial r} (rB) = \text{div}_{(r,z)} \left( -\frac{1}{2\mu_0} B^2, 0 \right) = -\frac{1}{2\mu_0} \frac{B^2}{r},\]
\[(j \times B)_\phi = 0,\]
\[(j \times B)_z = - \frac{B}{\mu_0} \frac{\partial B}{\partial z} = \text{div}_{(r,z)} \left( 0, -\frac{1}{2\mu_0} B^2 \right),\] (3.13)

where \(\mu_0\) denotes the magnetic permeability of the vacuum. After some calculations, involving in particular (3.13), we see that (3.6) can be written as a system of two scalar equations, namely
\[
\frac{\partial}{\partial t} (\rho v_r) + \text{div} f_{r, \text{invisc}} + \text{div} f_{r, \text{visc}} = q_r,\]
\[
\frac{\partial}{\partial t} (\rho v_z) + \text{div} f_{z, \text{invisc}} + \text{div} f_{z, \text{visc}} = 0,\] (3.14) (3.15)

with the flux functions
\[
f_{r, \text{invisc}} = \left( p_h + p_e + \frac{B^2}{2\mu_0} + \rho \nu_r^2 \right) e_r + \left( \rho \nu_r v_z \right) e_z, \quad f_{r, \text{visc}} = -\tau_{rr} e_r - \tau_{rz} e_z,\]
\[
f_{z, \text{invisc}} = \left( \rho \nu_r v_z \right) e_r + \left( p_h + p_e + \frac{B^2}{2\mu_0} + \rho \nu_z^2 \right) e_z, \quad f_{z, \text{visc}} = -\tau_{rz} e_r - \tau_{zz} e_z,\] (3.16)

containing the components of the viscous part of the stress tensor
\[
\tau_{rr} = \frac{2}{3} \mu_h \left( 2 \frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{\partial v_z}{\partial z} \right), \quad \tau_{rz} = \mu_h \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right), \quad \tau_{zz} = \frac{2}{3} \mu_h \left( 2 \frac{\partial v_z}{\partial z} - \frac{v_r}{r} - \frac{\partial v_r}{\partial r} \right),\] (3.17)

and the source term
\[
q_r = \frac{1}{r} \left( p_h + p_e - \frac{B^2}{2\mu_0} - 2\mu_h \left( \frac{v_r}{r} - \frac{1}{3} \text{div} \mathbf{v} \right) \right).\] (3.18)

### 3.1.3 Balance of energy

The heavy-particle energy \(E_h\) satisfies the conservation law
\[
\frac{\partial E_h}{\partial t} + \text{div} f_{h, \text{invisc}} + \text{div} f_{h, \text{visc}} = q_h,\] (3.19)

with the fluxes
\[
f_{h, \text{invisc}} = \left( E_h + p_h + p_e + \frac{B^2}{2\mu_0} \right) \mathbf{v},\]
\[
f_{h, \text{visc}} = \left( \tau_{rr} v_r + \tau_{rz} v_z + k_h \frac{\partial T_h}{\partial r} \right) e_r - \left( \tau_{rz} v_r + \tau_{zz} v_z + k_h \frac{\partial T_h}{\partial z} \right) e_z,\] (3.20)

and the source term
\[
q_h = \left( p_e + \frac{B^2}{2\mu_0} \right) \text{div} \mathbf{v} - \frac{B^2}{\mu_0 r} v_r + H_{\text{coll}},\] (3.21)

where \(H_{\text{coll}}\) defines a collision term defined by
\[
H_{\text{coll}} = n_e (T_e - T_h) \sum_{i=0}^{6} n_i \alpha_{ei},\] (3.22)

with \(\alpha_{ei}\) denoting the heat transfer coefficients, which depend on \(n_i\) and \(T_e\).

In addition to the partial differential equations, we require the (algebraic) constitutive state equations for the heavy-particle fluid
\[
p_h = k n_h T_h, \quad E_h = \frac{3}{2} k n_h T_h + \frac{1}{2} \rho |\mathbf{v}|^2,\] (3.23)
where $k$ is the Boltzmann constant. The coefficient $k_h$ coming up in the viscous term in (3.19) is the thermal conductivity of the heavy-particle fluid, depending on $n_i$, $T_h$, $T_e$. On the other hand, we have the constitutive relation (see, e.g., [37], p. 73)

$$\frac{\mu_h c_p}{k_h} = \Pr,$$

(3.24)

with the specific heat at constant pressure $c_p = \frac{5}{2} k$ and with the Prandtl number $\Pr$. For a monoatomic gas, as in our case, $\Pr = 2/3$, and, consequently,

$$k_h = \frac{15}{4} \frac{k}{m_h} \mu_h.$$  

(3.25)

This implies that also $\mu_h$ depends on $p_h$ and $n_i$ ($i = 0, \ldots, 6$).

### 3.2 Modelling of the electron flow

Let us denote by $v_e$ the velocity of the electron flow. Then the electron energy per unit volume, denoted by $E_e$, satisfies the balance equation

$$\frac{\partial E_e}{\partial t} = - \text{div}(E_e v_e) - p_e \text{div} v_e + \text{div}(k_e \nabla T_e) - \text{div} \left( \frac{3}{2} kT_e j_{D,e} \right) - \mathcal{H}_{\text{coll}} + \frac{|j|^2}{\sigma} - \sum_{i=0}^{5} \omega_{i+1} \chi_{i \rightarrow i+1},$$

(3.26)

where $k_e$ is the thermal conductivity of the electron flow, $\sigma$ is the electrical conductivity, $\chi_{i \rightarrow i+1}$ is the energy of ionization of $s_i$, and

$$j_{D,e} = \sum_{i=1}^{6} i j_{D,i}$$

(3.27)

is the diffusion current, depending on $n_i$, $T_h$, and $T_e$. The last term of eq. (3.26) as well as $k_e$ depend on $T_e$ and on the species densities $n_i$. If we neglect the electron inertia then the following equation holds:

$$j = e n_e (v - v_e).$$

(3.28)

This implies

$$- \text{div}(E_e v_e) - p_e \text{div} v_e = - \text{div}(E_e v) - p_e \text{div} v + \frac{5}{2} k \frac{1}{c_e} j \cdot \nabla T_e - \frac{1}{e n_e} j \cdot \nabla p_e.$$  

(3.29)

Replacing (3.27) and (3.29) in (3.26), we get

$$\frac{\partial E_e}{\partial t} = - \text{div}(E_e v) - p_e \text{div} v + \frac{5}{2} k \frac{1}{c_e} j \cdot \nabla T_e - \frac{1}{e n_e} j \cdot \nabla p_e - \text{div} \left( \frac{3}{2} kT_e \sum_{i=1}^{6} i j_{D,i} \right)$$

$$+ \text{div}(k_e \nabla T_e) + \frac{|j|^2}{\sigma} + n_e (T_h - T_e) \cdot \sum_{i=0}^{5} n_i \alpha_{ei} - \sum_{i=0}^{5} \omega_{i+1} \chi_{i \rightarrow i+1}.$$  

(3.30)

The following constitutive state equations are also associated with the electron flow:

$$p_e = kn_e T_e, \quad E_e = \frac{1}{2} kn_e T_e.$$  

(3.31)

For simplicity we assume that turbulence effects are negligible. For a more detailed description, including the precise dependence of different coefficients on the flow variables, we refer the reader to [26].

### 3.3 Modelling of the magnetic field

For the magnetic field we assume that the Maxwell equations are satisfied:

$$\text{rot} \, B = \mu_0 j, \quad \text{rot} \, E = - \frac{\partial B}{\partial t}, \quad \text{div} \, B = 0.$$  

(3.32)

In addition, we introduce Ohm’s law for plasmas:

$$E = \frac{j}{\sigma} - v \times B + \beta (j \times B) - \beta \nabla p_e,$$

(3.33)
As indicated before, in order to formulate the coupled problem associated with the MPD flow, we begin with simpler coupled models. In this section we investigate the case of 2D, compressible, one-phase fluid flows. Let \( \Omega \) be a given bounded, simply connected domain in \( \mathbb{R}^2 \). Therefore, the physical Dirichlet boundary conditions that we are going to use for \( B \), are just appropriate. Eq. (3.34) can also be expressed in terms of the stream function \( \Psi = rB \).

To summarize, we consider the following basic unknowns:

\[
\begin{align*}
  u_i & \quad (i = 0, \ldots, 6), \\
  v & = v_r e_r + v_z e_z, \\
  E_h, & \quad E_z, \quad B,
\end{align*}
\]

depending on \((r, z, t)\) (i.e., 12 scalar unknowns), which satisfy the PDE system (3.1), (3.14)–(3.19), (3.30), (3.34). This is a closed system due to the state equations given by (3.23) and (3.31).

In order to formulate the mathematical MPD model completely, we have to add to the PDE system appropriate initial and boundary conditions, such that the resulting model is well-posed and consistent with physics. This is a rather ambitious goal. Indeed, according to our engineering application, different boundary conditions must be assigned to different parts of the boundary. Thus, the resulting model is extremely complex and no complete mathematical theory is available yet for such a boundary value problem. However, we can use arguments from physics to establish appropriate boundary conditions. Actually, even if the complete model is established, we need numerical solutions and the computational cost becomes tremendous for such a model. Fortunately, in a large subdomain of \( \Omega \) we may use simplified models to describe the plasma flow. Indeed, far from the two electrodes, in the far-field subdomain associated with the test tank, the magnetic field can be considered completely absent and also both, the viscosity effect and the heat conductivity of the heavy-particle flow are strongly dominated by the convective (Euler) part. For the corresponding simplified PDE-system, different specific boundary conditions should be associated with the corresponding parts of the outer boundary. In the small near field we consider the full, original PDE system. So we arrive at a coupled problem which is more feasible for numerical simulations. Then a new difficulty arises; i.e., to establish appropriate transmission conditions at the fictitious coupling boundary (interface) \( \Gamma \) separating the two subdomains, in such a way that the coupled model results in a well-posed problem in the sense of Hadamard. For a numerical treatment of the compressible plasma flow with 3 heavy-particle species \((\text{Ar}^{0}, \text{Ar}^{1+}, \text{Ar}^{2+})\) in chemical equilibrium by means of heterogeneous domain decomposition methods we refer the reader to our previous papers [2,7–9].

In what follows we shall formulate such a coupled problem in two spatial dimensions and shall see that it is consistent with the physics. Our approach for investigating the interface conditions is based on singular perturbation theory, which seems here to be most appropriate. In order to formulate the coupled problem associated with our MPD flow, we first shall develop our analysis for simpler models of compressible fluid flows. Indeed, due to the high complexity of our MPD model, we have to derive some facts from simpler situations. On the other hand, the particular models we are going to investigate are interesting by themselves and can be applied to further engineering problems, too. Since 1999 we have investigated 1D-coupled problems (see [3,15–17]). Specifically, we have considered coupled hyperbolic/elliptic problems with a transmission condition based on the conservation of flux. Such a coupled problem is viewed as a degenerate case of an elliptic/elliptic problem obtained by introducing an artificial viscosity coefficient \( \varepsilon > 0 \) in the hyperbolic subdomain. By performing a singular perturbation analysis of this perturbed problem, we are able to correct the solution of the original coupled problem by an internal boundary layer term associated with the hyperbolic subdomain. This is an extension and improvement of the analysis in [24]. For this 1D case, a rigorous treatment is possible and our numerical results are in good agreement with the theoretical analysis. We do not insist more on this case, although it seems a good foundation for further investigations of transmission conditions.

In two spatial dimensions, the coupling is formulated for the classical Navier-Stokes and Euler equations, i.e., a parabolic/hyperbolic coupling. Again, asymptotic analysis and singular perturbation are used for the construction of appropriate transmission conditions.

### 4 Asymptotic analysis of some 2D Cartesian coupled models for compressible one-phase fluid flows

As indicated before, in order to formulate the coupled problem associated with the MPD flow, we begin with simpler coupled models. In this section we investigate the case of 2D, compressible, one-phase fluid flows. Let \( \Omega \) be a given bounded, simply connected domain in \( \mathbb{R}^2 \). The case when \( \Omega \) is unbounded can be treated similarly.
We assume that \( \Omega \) is decomposed by a line segment \( \Gamma \) into two disjoint subdomains \( \Omega^- \) and \( \Omega^+ \). In fact, our results extend to a general smooth interface \( \Gamma \), but the computation is much simpler if \( \Gamma \) is a line segment. We assume that \( \Gamma \) is included in the \( x_2 \)-axis, \( \Gamma := \Omega \cap \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \} \), separating the subdomains \( \Omega^- := \{ x \in \Omega : x_1 < 0 \} \) and \( \Omega^+ := \{ x \in \Omega : x_1 > 0 \} \).

Before introducing the coupled problem associated with this decomposition of \( \Omega \), we first discuss some aspects of 2D compressible Navier-Stokes flows.

### 4.1 2D compressible one-phase Navier-Stokes models

We consider the conservation laws for mass, momentum and energy associated with such a fluid, written in Cartesian conservative form:

\[
\frac{\partial w}{\partial t} - \text{div} R(w, \nabla w, \nabla T) + \text{div} F(w) = 0 \quad \text{in} \quad \Omega \times (0, t_1).
\] (4.1)

Here, \((0, t_1)\) is some time interval; \( w = (\rho, \rho v_1, \rho v_2, E)^\top \) collects the conservative variables, with \( \rho \) the mass density, \( v = (v_1, v_2)^\top = v_1 e_1 + v_2 e_2 \) the velocity vector, and \( E \) the total energy. The function \( F = (F_1, F_2) \) contains the inviscid fluxes, which are given by

\[
F_1(w) = \begin{pmatrix}
\rho v_1 \\
\rho v_1^2 + p \\
\rho v_1 v_2 \\
(E + p) v_1
\end{pmatrix}, \quad F_2(w) = \begin{pmatrix}
\rho v_2 \\
\rho v_1 v_2 \\
\rho v_2^2 + p \\
(E + p) v_2
\end{pmatrix},
\]

including the pressure \( p \). The dissipative terms \( R = (R_1, R_2) \) are defined by

\[
R_1 = \begin{pmatrix}
0 \\
\tau_{11} \\
\tau_{12} \\
\tau_{11} v_1 + \tau_{12} v_2 + k \frac{\partial T}{\partial x_1}
\end{pmatrix}, \quad R_2 = \begin{pmatrix}
0 \\
\tau_{21} \\
\tau_{22} \\
\tau_{21} v_1 + \tau_{22} v_2 + k \frac{\partial T}{\partial x_2}
\end{pmatrix},
\]

with the absolute temperature \( T \) and with \( \tau_{ij} \) denoting the components of the viscous part of the stress tensor (see (3.7)):

\[
\tau_{ij} = \lambda \text{div} v \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{for} \quad i, j = 1, 2.
\]

Assuming that the viscosity coefficients satisfy the thermodynamical relation \( 3\lambda + 2\mu = 0 \), we can write \( \tau_{ij} \) in terms of \( \mu \) only:

\[
\tau_{11} = \mu \left( \frac{4}{3} \frac{\partial v_1}{\partial x_1} - \frac{2}{3} \frac{\partial v_2}{\partial x_2} \right), \quad \tau_{12} = \tau_{21} = \mu \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \quad \tau_{22} = \mu \left( -\frac{2}{3} \frac{\partial v_1}{\partial x_1} + \frac{4}{3} \frac{\partial v_2}{\partial x_2} \right).
\] (4.4)

In (4.3) \( k \) represents the thermal conductivity, which is connected to \( \mu \) by means of an equation of type (3.24):

\[
k = \frac{c_p}{\Pr} \mu.
\] (4.5)
We also employ the state equations

\[ p = R \rho T \quad \text{and} \quad E = \rho \left( c_V T + \frac{|v|^2}{2} \right) \]  

(4.6)

with the gas constant \( R \) and the specific heat at constant volume \( c_V \). These are related by the equation \( R = (\gamma - 1) c_V \), where \( \gamma \) is the Poisson adiabatic constant.

Note that external forces such as gravity, electromagnetic forces, etc., as well as heat sources are considered absent, although their presence would not significantly change our analysis.

Note also that in general \( k \) and \( \mu \) depend smoothly on \( \rho \) and \( T \).

Due to (4.6), the Navier-Stokes system (4.1) is closed. Indeed, using (4.6) we can write (4.1) in terms of the physical variables \( z = (\rho, v_1, v_2, T) \). In particular, the energy equation (4.1) can be written as

\[ \rho c_V \left( \frac{\partial T}{\partial t} + v_1 \frac{\partial T}{\partial x_1} + v_2 \frac{\partial T}{\partial x_2} \right) + R \rho T \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) = \tau_{11} \frac{\partial v_1}{\partial x_1} + \tau_{12} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + \tau_{22} \frac{\partial v_2}{\partial x_2} + \text{div}(k \nabla T). \]  

(4.7)

The whole Navier-Stokes system consists of three parabolic equations, whereas the mass equation (4.1) is hyperbolic in \( \rho \). When the fluid is inviscid, i.e. \( \mu = 0 \) (and, hence, \( k = 0 \)), we obtain the Euler system which is purely hyperbolic (this case will be discussed later, in Sect. 4.2).

At present, there are only partial existence results available for initial boundary value problems associated with the Navier-Stokes system. We refer the reader to the monograph [32] by P.L. Lions for a detailed presentation of recent advances (see also E. Feireisl [20, 21]). In these works are some results stating the existence of global weak solutions in the case \( p = p(\rho) \) (barotropic fluid), in particular for \( p = a\rho^\gamma \) (\( \gamma > 1 \), \( a > 0 \)), where the energy equation is not necessary since the system consisting of the continuity equation and the balance of momentum there is already a closed system. If \( \Omega \) is a bounded domain then the following no-slip boundary condition is mostly used:

\[ v = 0 \quad \text{on} \quad \partial \Omega \times (0, t_1). \]  

(4.8)

In addition, initial conditions at \( t = 0 \) for \( \rho \) and \( v \) are also needed.

Some special cases of compressible models with temperature are also studied in [32] and the existence of global weak solutions is proved. It is quite evident that mathematically much remains to be done. However, we shall here presume that for smooth domains and regular data our Navier-Stokes models have unique global smooth solutions, depending continuously on the data, i.e. well-posedness in Hadamard’s sense. Moreover, we assume well-posedness for Navier-Stokes models with mixed boundary conditions that are physically relevant. For example, let us consider the case that \( S \) is a smooth bounded domain, then we shall assume that the full Navier-Stokes system complemented by (4.9), (4.10) and by initial conditions, has a unique global smooth solution in \( \Omega \). Possible discontinuities are expected to appear at some points of \( S_1 \cap S_2 \).

Other cases of mixed boundary conditions can be discussed similarly. For lack of mathematical information, we have to take into account some physical motivation. In particular, note that one of the following three boundary conditions can be used for the temperature:

- \( T = T_{\text{wall}} \) (fixed wall temperature; Dirichlet condition);
- \( k \frac{\partial T}{\partial n} = q \) (fixed heat flux; Neumann condition);
- \( k \frac{\partial T}{\partial n} = \alpha (T - T_{\text{wall}}) \) (heat flux proportional to local heat transfer).
4.2 2D compressible, one-phase Euler models

The inviscid flow equations of the Euler system are obtained from (4.1) (or from the corresponding system in 2D case. The 2D Euler system can be written in quasilinear form with respect to the primitive variables require as many boundary conditions as there are characteristics entering the domain \[23\]. Let us briefly explain this for the

\[\Omega=\mathbb{R}^N, \quad N \geq 1,\]

where \(\Omega\) is the computational domain. The case of an open boundary is even more delicate. However, it is straightforward to construct characteristics and then to require as many boundary conditions as there are characteristics entering the domain \[23\].

Then the characteristic equation reads

\[
\det (\lambda I - (n_1 A + n_2 B)) = 0,
\]

where

\[
A = \begin{pmatrix} v_1 & \rho & 0 & 0 \\ 0 & v_1 & 0 & 1/\rho \\ 0 & 0 & v_1 & 0 \\ 0 & \gamma p & 0 & v_1 \end{pmatrix}, \quad B = \begin{pmatrix} v_2 & 0 & \rho & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & v_2 & 1/\rho \\ 0 & 0 & \gamma p & v_2 \end{pmatrix}.
\]

The characteristic variables (Riemann invariants) corresponding to \(\lambda_i(n) (i = 1, \ldots, 4)\) are given by

\[
U_1 = (v \cdot \tau) (x, t), \quad U_2 = S_0 + c_F \log \left( \frac{p/p_0}{(\rho/\rho_0)^\gamma} \right) (x, t),
\]

\[
U_3 = \left( v \cdot n + \frac{2a}{\gamma - 1} \right) (x, t), \quad U_4 = \left( -v \cdot n + \frac{2a}{\gamma - 1} \right) (x, t),
\]

with \(n\) the unit normal and \(\tau\) the tangential vector (see the figure above). Hence, different local boundary conditions are obtained by prescribing at different points \(M \in \partial \Omega\) those quantities \(U_i\) which correspond to negative eigenvalues \(\lambda_i\). For example, if we have outflow at \(M\) (i.e., \(v \cdot n > 0\)), then \(U_4\) should be prescribed when this outflow is subsonic (i.e., \(v \cdot n - a < 0\)); and no characteristic variable is prescribed when the outflow is supersonic (i.e., \(v \cdot n - a > 0\)). The physical explanation for
such boundary conditions is not too obvious, but they generate well-posed problems. The number of boundary conditions at points of an open boundary given by the method of characteristics is the same as that found by J. Oliger and A. Sundström [34] (see also [23], p. 40), who are using the so-called energy method. They formulate several different combinations of boundary conditions, associated with inflow or outflow parts of the boundary, leading to well-posed models. For example, in the case of a subsonic outflow, only one quantity should be prescribed and, in particular, this can be the normal component of the velocity or the pressure. This fact will be used later on for our MPD model, since we have a subsonic outflow associated with the pump extracting plasma from the tank.

4.3 Transmission conditions for Cartesian, coupled 2D models associated with compressible flows

We return to the bounded domain \( \Omega \subset \mathbb{R}^2 \) which was divided into two subdomains, \( \Omega^- \) and \( \Omega^+ \) by the line segment \( \Gamma \). Now we consider the full Cartesian Navier-Stokes system in \( \Omega^+ \) and the simplified Euler system in \( \Omega^- \). From the preceding sections we know how to associate outer boundary conditions with both these PDE systems. The boundary conditions will here be expressed by means of general boundary operators denoted by \( B_{\text{visc}} \) and \( B_{\text{invisc}} \). Now, the question is how to establish appropriate transmission conditions at the interface \( \Gamma \) separating the two subdomains in such a way that the corresponding coupled Euler/Navier-Stokes problem is still well-posed and consistent with physical principles. Clearly, the equations expressing the conservation of flux should be used as transmission conditions. However, in order to define the coupled problem completely, one usually employs additional transmission conditions expressing the continuity of the flow variables. An approach often used is based on the method of characteristics described in the previous section, since in \( \Omega^- \) we require the Euler system [18,25,35]. The method of characteristics is very appropriate for establishing boundary conditions associated with the Euler system, corresponding to open (free-stream) parts of the outer boundary. In order to establish transmission conditions at the artificial interface \( \Gamma \) we shall use a different approach (see [10, 11]), which is based on singular perturbation theory. In fact, this approach is again very natural and consistent with physical principles. For instance, the flux conservation equations are re-obtained; and the Mach number will appear as a key parameter.

Now, let us consider the perturbed viscous/viscous coupled problem, say \( (P_2) \), obtained by employing the full Navier-Stokes system in \( \Omega^+ \) and the same system with \( \mu = k = \varepsilon \) in \( \Omega^- \), where \( \varepsilon > 0 \) is a small parameter. The subdomain \( \Omega^- \) plays here the rôle of the far field, where the viscosity and the heat conduction both are negligible. The problem \( (P_2) \) is supplemented by initial conditions, appropriate outer boundary conditions, as well as natural transmission conditions expressing the conservation of flux and the continuity of all the flow variables, i.e.,

\[
\begin{align*}
\mathbf{w}^- &= \mathbf{w}^+ \quad \text{on} \quad \Gamma \times (0, t_1), \\
-R^{-}(\mathbf{w}^-, \nabla \mathbf{w}^-, \nabla T^-) \cdot \mathbf{n} + \mathbf{F}(\mathbf{w}^-) \cdot \mathbf{n} &= -R^{+}(\mathbf{w}^+, \nabla \mathbf{w}^+, \nabla T^+) \cdot \mathbf{n} + \mathbf{F}(\mathbf{w}^+) \cdot \mathbf{n} \quad \text{on} \quad \Gamma \times (0, t_1),
\end{align*}
\]

where \( \mathbf{w}^\pm \) represent the restrictions of the solution \( \mathbf{w} \) to the subdomains \( \Omega^\pm \), respectively, and \( \mathbf{R}^\pm \) collects the viscous fluxes corresponding to \( \mu = k = \varepsilon \) (see Sect. 4.1). Of course, \( \mathbf{w}^\pm \) depend on \( \varepsilon \): \( \mathbf{w}^\pm = \mathbf{w}^\pm_\varepsilon \). In our special situation we have \( \mathbf{n} = (1, 0)^T \). Therefore, the transmission conditions take a special form.

The viscous/viscous transmission-boundary value problem \( (P_2) \):

Find the pair \( \mathbf{w}_\varepsilon = (\mathbf{w}^-_\varepsilon, \mathbf{w}^+_\varepsilon) : (\Omega^-, \Omega^+) \times (0, t_1) \to \mathbb{R}^4 \) such that the equations

\[
\begin{align*}
\frac{\partial \mathbf{w}^-_\varepsilon}{\partial t} - \text{div} \mathbf{R}^- (\mathbf{w}^-_\varepsilon, \nabla \mathbf{w}^-_\varepsilon, \nabla T^-) + \text{div} \mathbf{F}(\mathbf{w}^-_\varepsilon) &= 0 \quad \text{in} \quad \Omega^- \times (0, t_1), \\
\frac{\partial \mathbf{w}^+_\varepsilon}{\partial t} - \text{div} \mathbf{R}^+ (\mathbf{w}^+_\varepsilon, \nabla \mathbf{w}^+_\varepsilon, \nabla T^+) + \text{div} \mathbf{F}(\mathbf{w}^+_\varepsilon) &= 0 \quad \text{in} \quad \Omega^+ \times (0, t_1)
\end{align*}
\]

are satisfied together with the initial conditions

\[
\mathbf{w}^\pm(x, 0) = (\mathbf{w}^\pm_0 := \mathbf{w}_0|_{\Omega^\pm})(x) \quad \text{for} \quad x \in \Omega^\pm, \quad (4.18)
\]

the boundary conditions

\[
B^\pm_{\text{visc}}(\mathbf{w}^\pm_\varepsilon) = 0 \quad \text{on} \quad (\partial \Omega^\pm \cap \partial \Omega) \times (0, t_1), \quad (4.19)
\]

and the natural transmission conditions

\[
\begin{align*}
\mathbf{w}^+_\varepsilon &= \mathbf{w}^-_\varepsilon \quad \text{on} \quad \Gamma \times (0, t_1), \\
-R_1(\mathbf{w}^+_\varepsilon, \nabla \mathbf{w}^+_\varepsilon, \nabla T^+) + \mathbf{F}_1(\mathbf{w}^+_\varepsilon) &= -R_1(\mathbf{w}^-_\varepsilon, \nabla \mathbf{w}^-_\varepsilon, \nabla T^-) + \mathbf{F}_1(\mathbf{w}^-_\varepsilon) \quad \text{on} \quad \Gamma \times (0, t_1).
\end{align*}
\]

In general, problem \( (P_2) \) is singularly perturbed with respect to the norm of uniform convergence (see [19], Chap. 1). Indeed, assuming that \( \mathbf{w}^-_\varepsilon \) converges uniformly in \( \overline{\Omega} \times [0, t_1] \) to some \( \mathbf{w} = (\mathbf{w}^-, \mathbf{w}^+) \), then \( \mathbf{w}^+ \) should satisfy an overdetermined
boundary value problem in $\Omega^+ \ (\text{cf. (4.20), (4.21)}).$ Hence, a boundary (transition) layer phenomenon might occur at the interface $\Gamma$. Therefore, $\Gamma$ or a part of it might be a transition layer.

For more information on singular perturbation theory we refer the reader to the books [3, 19, 29–31, 33, 36].

In principle, if we set $\varepsilon = 0$ in $(P_2)$ we expect to get a reduced problem $(P_0)$ which describes an Euler / Navier-Stokes coupling, whose solution is the first regular term of the asymptotic expansion of the solution of $(P_2)$. We can show by formal arguments that this indeed is the case, but some of the transmission conditions of $(P_2)$ cannot be maintained for $(P_0)$ on some part of $\Gamma$ where a boundary layer is present. Actually, we shall see that the boundary layer can appear only on the inviscid side of $\Gamma$. It is expected that the outer boundary conditions of $(P_2)$ corresponding to $\partial \Omega \cap \partial \Omega^+$ are maintained for $(P_0)$, however we have a different situation at $\partial \Omega \cap \partial \Omega^-$. Here, for physical reasons, “inviscid” boundary conditions are appropriate for $(P_0)$, instead of the original “viscous” boundary conditions. For example, if $v_\varepsilon = 0$ and $T_\varepsilon = T_{\text{wall}}$ are satisfied on some part of $\partial \Omega \cap \partial \Omega^-$, the so-called wall, then $(P_0)$ is there equipped with the slip condition $v \cdot n = 0$.

In our asymptotic analysis, we replace the conservative variables $w = (\rho, \rho v_1, \rho v_2, \varepsilon)^T$ by the physical variables $z = (\rho, v_1, v_2, T)^T$, i.e.

$$z_1 = w_1, \quad z_2 = \frac{w_2}{w_1}, \quad z_3 = \frac{w_3}{w_1}, \quad \text{and} \quad z_4 = \frac{1}{c_v} \left( \frac{w_4}{w_1} - \frac{w_2^2 + w_3^2}{2w_1^2} \right).$$

Hence, the solution of the “regularized” problem $(P_2)$ will be denoted by

$$z_\varepsilon = (z^-_\varepsilon, z^+_\varepsilon), \quad \text{with} \quad z^\pm_\varepsilon = (\rho^\pm_\varepsilon, v^1_\varepsilon, v^2_\varepsilon, T^\pm_\varepsilon).$$

The solution $z_\varepsilon$ is sought in a two-scaled asymptotic form:

$$z^-_\varepsilon(x, t) = z^-(x, t) + c(\tau, x_2, t) + r^-_\varepsilon(x, t) \quad \text{for} \quad (x, t) \in \Omega^- \times (0, t_1),$$

$$z^+_\varepsilon(x, t) = z^+(x, t) + c(\tilde{\tau}, x_2, t) + r^+_\varepsilon(x, t) \quad \text{for} \quad (x, t) \in \Omega^+ \times (0, t_1),$$

(4.22)

where $\tau = -x_1/\varepsilon$, $x_1 < 0$ and $\tilde{\tau} = x_1/\varepsilon$, $x_1 > 0$ are the rapid variables associated with the two sides of $\Gamma$. The vector-valued function $z = (z^-, z^+)$ is the zeroth order term of the regular expansion corresponding to $z_\varepsilon$, whereas $(c, \tilde{c})$ is the zeroth order term of the expansion expansion corresponding to $z_\varepsilon$; and $r_\varepsilon = (r^-_\varepsilon, r^+_\varepsilon)$ represents the remainder of order zero. Our purpose is to derive the problem $(P_0)$ satisfied by $z = (z^-, z^+)$, $z^\pm = (\rho^\pm, v^1_\varepsilon, v^2_\varepsilon, T^\pm)$ and to find out as much as possible for the corrections. We insert $z_\varepsilon$ given by (4.22) into $(P_2)$ and use the formulas

$$\frac{\partial}{\partial x_1} = -\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} \quad \text{for} \quad x_1 < 0 \quad \text{and} \quad \frac{\partial}{\partial x_1} = \frac{1}{\varepsilon} \frac{\partial}{\partial \tau} \quad \text{for} \quad x_1 > 0.$$ 

Then we equate the coefficients of like powers of $\varepsilon$ and separate those depending on $\tau$ and $\tilde{\tau}$ from the other terms which are independent of $\tau$ and $\tilde{\tau}$. After some tedious computations (see [11]) we can show that $z = (z^-, z^+)$ (or, equivalently, $w = (w^-, w^+)$) satisfies the following problem:

The inviscid/viscous coupled transmission problem $(P_0)$:

$$\frac{\partial w^-}{\partial t} + \text{div} F(w^-) = 0 \quad \text{in} \quad \Omega^- \times (0, t_1),$$

$$\frac{\partial w^+}{\partial t} - \text{div} R(w^+, \nabla w^+, \nabla T^+) + \text{div} F(w^+) = 0 \quad \text{in} \quad \Omega^+ \times (0, t_1),$$

(4.23)

$$w^\pm(x, 0) = w^\pm_0(x) \quad \text{for} \quad x \in \Omega^\pm,$$

$$\mathcal{B}^-_{\text{visc}}(w^-) = 0 \quad \text{on} \quad (\partial \Omega \cap \partial \Omega^+) \times (0, t_1),$$

$$\mathcal{B}^+_{\text{visc}}(w^-) = 0 \quad \text{on} \quad (\partial \Omega \cap \partial \Omega^-) \times (0, t_1),$$

$$\mathcal{F}_1(w^-) = -R_1(w^+, \nabla w^+, \nabla T^+) + \mathcal{F}_1(w^+)$$

on $\Gamma \times (0, t_1).$

(4.24)

(4.25)

(4.26)

(4.27)

Note that the last equation expresses the conservation of flux for the inviscid/viscous flow, which is a traditionally used condition [18, 25, 35]. Problem $(P_0)$ should be supplemented by transmission conditions expressing the continuity of the flow variables, at least on some part of $\Gamma$. These continuity conditions can be stated precisely by investigating the corrections, i.e. the boundary layer functions. Actually, we can show that $\tilde{c} \equiv 0$ which means that a boundary layer may appear only on the inviscid side of $\Gamma$. The problem satisfied by $c(\tau, x_2, t) = (c_1, c_2, c_3, c_4)^T(\tau, x_2, t)$, say $(P_2)$, can explicitly be formulated as a system of ordinary differential equations, namely:

$$\frac{4}{3} \frac{\partial^2 c_2}{\partial \tau^2} + (\rho^-_\varepsilon + c_1) \frac{\partial c_2}{\partial \tau} + R(T^-_\varepsilon + c_4) \frac{\partial c_1}{\partial \tau} + R(\rho^-_\varepsilon + c_1) \frac{\partial c_4}{\partial \tau} = 0,$$

$$\frac{\partial c_1}{\partial \tau} + (\rho^-_\varepsilon + c_1) \frac{\partial c_1}{\partial \tau} = 0,$$

$$\frac{\partial c_2}{\partial \tau} + (\rho^-_\varepsilon + c_1)(v_{1\varepsilon} + c_2) \frac{\partial c_2}{\partial \tau} + R(T^-_\varepsilon + c_4) \frac{\partial c_1}{\partial \tau} + R(\rho^-_\varepsilon + c_1) \frac{\partial c_4}{\partial \tau} = 0,$$

$$\frac{\partial c_3}{\partial \tau} + (\rho^-_\varepsilon + c_1)(v_{2\varepsilon} + c_3) \frac{\partial c_3}{\partial \tau} + R(T^-_\varepsilon + c_4) \frac{\partial c_1}{\partial \tau} + R(\rho^-_\varepsilon + c_1) \frac{\partial c_4}{\partial \tau} = 0,$$

$$\frac{\partial c_4}{\partial \tau} + (\rho^-_\varepsilon + c_1)(v_{2\varepsilon} + c_4) \frac{\partial c_4}{\partial \tau} + R(T^-_\varepsilon + c_4) \frac{\partial c_1}{\partial \tau} + R(\rho^-_\varepsilon + c_1) \frac{\partial c_4}{\partial \tau} = 0.$$
\[ \frac{\partial^2 c_3}{\partial \tau^2} + (\rho_\Gamma^- + c_1)(v_{1,\Gamma}^- + c_2) \frac{\partial c_3}{\partial \tau} = 0, \]
\[ \frac{\partial^2 c_4}{\partial \tau^2} + 4 \left( \frac{\partial c_2}{\partial \tau} \right)^2 + \left( \frac{\partial c_3}{\partial \tau} \right)^2 + R(\rho_\Gamma^- + c_1)(T_{\Gamma}^- + c_4) \frac{\partial c_2}{\partial \tau} + c_V(\rho_\Gamma^- + c_1)(v_{1,\Gamma}^- + c_2) \frac{\partial c_4}{\partial \tau} = 0, \]

together with the initial conditions
\[ c_1(0, x_2, t) = (\rho_\Gamma^+ - \rho_\Gamma^-)(x_2, t), \]
\[ c_{i+1}(0, x_2, t) = (v_{i,\Gamma}^+ - v_{i,\Gamma}^-)(x_2, t) \quad (i = 1, 2), \]
\[ c_4(0, x_2, t) = (T_\Gamma^+ - T_\Gamma^-)(x_2, t) \quad \text{for} \quad (0, x_2, t) \in \Gamma \times (0, t_1), \]

and the conditions at infinity
\[ \lim_{\tau \to \infty} c_i(\tau, x_2, t) = 0 \quad \text{for} \quad i = 1, \ldots, 4. \]

Here we denote \( \rho_\Gamma^\pm(x_2, t) := \rho^\pm(0, x_2, t), v_1,\Gamma^\pm(x_2, t) = v_1,\Gamma^\pm(0, x_2, t) \quad (i = 1, 2) \) and \( T_\Gamma^\pm(x_2, t) := T^\pm(0, x_2, t) \) for \( (0, x_2, t) \in \Gamma \times (0, t_1). \) Actually, \( x_2 \) and \( t \) can be omitted since here the principal independent variable is \( \tau. \) The last conditions are the requirements for the corrections, expressing the fact that they are negligible outside the transition layer. Here, problem \( (P_e) \) reduces to the solution of a Cauchy problem for a system of ordinary differential equations for \( c_2, c_4 \) with additional conditions for \( \tau \to \infty. \) For \( v_{1,\Gamma}^\neq \neq 0, \) which is a critical case, this system can be linearized about the origin:

\[ \frac{\partial}{\partial \tau} \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = \mathbf{A} \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} + \mathbf{B} \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} + (0, f(\tau))^T, \quad (4.28) \]

where \( \mathbf{A} = (a_{ij}) \) is a \( 2 \times 2 \) coefficient matrix with
\[ a_{11} = -(3/4)\rho_\Gamma^+ (v_{1,\Gamma}^- - R T_{\Gamma}^- / v_{1,\Gamma}^-), \quad a_{12} = -(3/4)R \rho_\Gamma^+, \quad a_{21} = -R \rho_\Gamma^- T_{\Gamma}^-, \quad a_{22} = -c_V \rho_\Gamma^- v_{1,\Gamma}^- \]

and \( \mathbf{B} \) represents a nonlinear function vanishing at the origin. Finally, \( f(\tau) = 0 \) for \( v_{1,\Gamma}^- < 0, \) and
\[ f(\tau) = (1/2)\rho_\Gamma^- v_{1,\Gamma}^- (v_{2,\Gamma}^+ - v_{2,\Gamma}^-)^2 e^{-2\rho_\Gamma^- v_{1,\Gamma}^+ \tau} \quad \text{if} \quad v_{1,\Gamma}^- > 0. \]

Denote by \( M_1 \) the **Mach number** associated with the normal velocity \( v_{1,\Gamma}^- \),
\[ M_1 = \frac{|v_{1,\Gamma}^-|}{\left( \frac{\gamma}{\gamma - 1} \frac{\rho_\Gamma^-}{\rho_\Gamma^+} \right)^{1/2}}, \]

where \( \rho_\Gamma^- \) denotes the pressure at the “inviscid” side of \( \Gamma, \) given by \( \rho_\Gamma^- = R \rho_\Gamma^- T_{\Gamma}^- \).

**Proposition 4.1** The eigenvalues \( \omega_1 \) and \( \omega_2 \) of the matrix \( \mathbf{A} \) are real and distinct. In addition, the following implications hold:

- (a) \( M_1 > 1 \) and \( v_{1,\Gamma}^- > 0 \) \( \Rightarrow \) \( \omega_i < 0 \) for \( i = 1, 2; \)
- (b) \( M_1 > 1 \) and \( v_{1,\Gamma}^- < 0 \) \( \Rightarrow \) \( \omega_i > 0 \) for \( i = 1, 2; \)
- (c) \( M_1 = 1 \) and \( v_{1,\Gamma}^- > 0 \) \( \Rightarrow \) \( \omega_1 = 0 \) and \( \omega_2 < 0; \)
- (d) \( M_1 = 1 \) and \( v_{1,\Gamma}^- < 0 \) \( \Rightarrow \) \( \omega_1 = 0 \) and \( \omega_2 > 0; \)
- (e) \( 0 < M_1 < 1 \) \( \Rightarrow \) \( \omega_1 > 0 \) and \( \omega_2 < 0. \)

The justification of this proposition involves some particular physical facts, such as \( \rho_\Gamma^- > 0. \)

On the basis of Proposition 4.1, the following two results can be proved for a **locally supersonic** inviscid/viscous flow across \( \Gamma \) (i.e., \( M_1 > 1 \)).

**Theorem 4.1** Assume that \( M_1 > 1 \) and \( v_{1,\Gamma}^- > 0 \). Then there exists some positive constant \( \delta > 0 \) such that for \( |v_{1,\Gamma}^- - v_{i,\Gamma}^-| < \delta \) \( (i = 1, 2) \) and \( |T_{\Gamma}^+ - T_{\Gamma}^-| < \delta, \) the problem \((P_e)\) has a unique solution \( c(\tau, x_2, t) \) that decays exponentially for \( \tau \to \infty, \) i.e.
\[ ||c(\tau, x_2, t)||_{\mathcal{S}} \leq K \cdot e^{-\alpha \tau} \quad \text{for all} \quad \tau \geq 0, \quad (4.29) \]

and where \( K, \alpha \) are positive constants depending on \( \delta \) and \( z, \) respectively, on \( w. \)

**Theorem 4.2** Assume that \( M_1 > 1 \) and \( v_{1,\Gamma}^- < 0. \) Then necessarily \( c_i = 0 \) for \( i = 1, 2, 3, 4, \) and, hence, the following conditions are satisfied:
\[ \rho_\Gamma^\pm = \rho_\Gamma^\pm, \quad v_{1,\Gamma}^\pm = v_{1,\Gamma}^\pm, \quad v_{2,\Gamma}^\pm = v_{2,\Gamma}^\pm, \quad T_{\Gamma}^\pm = T_{\Gamma}^\pm. \quad (4.30) \]
This conjecture has been checked by numerical tests, see [11]. In other words, in the case of a local subsonic inviscid/viscous conditions for \((\mathbf{P}_0)\), the approximate solution needs to be imposed at the interface, and the only transmission conditions for \((\mathbf{P}_0)\) are those expressing the conservation of flux across \(\Gamma\).

Theorem 4.1 states that, in the case of a locally supersonic flow leaving the “inviscid” Euler subdomain \(\Omega^-\), the transition layer correction \(c\) exists, hence can take care of possible local jumps in the components of the viscous/inviscid coupled solution which in this case are allowed at \(\Gamma\). The exponential decay (4.29) can be observed also numerically.

Fig. 6 shows the approximate solution \((c_2, c_4)\) of the system (4.28) with flow quantities \((\hat{\nu}_1, \nu_{1,\Gamma} > 0, \nu_{2,\Gamma}, T_{\Gamma}^{-1})^\top\) leading to Mach numbers \(M_1 = \nu_{1,\Gamma} (\gamma RT_{\Gamma}^{-1})^{-1/2} > 1\). The exponential decay of \((c_2, c_4)\) as \(\tau \to \infty\) implies also \(c_1(\tau), c_3(\tau) \to 0\) for \(\tau \to \infty\). Since in this case, the viscous/inviscid coupled solution may admit local discontinuities at \(\Gamma\) no continuity condition needs to be imposed at the interface, and the only transmission conditions for \((\mathbf{P}_0)\) are those expressing the conservation of flux across \(\Gamma\).

On the contrary, if \(M_1 > 1\), but \(\nu_{1,\Gamma} < 0\), i.e. in the case of a locally supersonic compressible flow entering the hyperbolic Euler subdomain \(\Omega^-\), one needs to add the continuity conditions (4.30) to problem \((\mathbf{P}_0)\) (Theorem 4.2). The corresponding numerical tests presented in [11] show that the approximate solution \((c_2, c_4)\) tends to infinity if \((c_2, c_4)(0, x_2, t) \neq (0, 0)\), hence this solution does not satisfy the condition \(c_{2,4}(\infty, x_2, t) = 0\). Therefore, in this case, the components \(c_2\) and \(c_4\) must vanish identically. Then the correction vector \(c\) vanishes identically, and, consequently, now the continuity conditions (4.30) have to be imposed at \(\Gamma\).

For the subsonic case \(M_1 < 1\), we could not yet prove corresponding results, but taking into account that the matrix \(A\) has a positive eigenvalue as well as the special form of system (4.28), we may conjecture that probably all \(c_i\) are necessarily zero. This conjecture has been checked by numerical tests, see [11]. In other words, in the case of a local subsonic inviscid/viscous flow across the artificial interface \(\Gamma\), i.e. \(M_1 < 1\), the continuity of all the flow variables should be employed as transmission conditions for \((\mathbf{P}_0)\). The same situation appears if \(M_1 = 1\) (local sonic flow) and \(\nu_{1,\Gamma} < 0\), as it can be seen in Fig. 7.

The case \(M_1 = 1\) and \(\nu_{1,\Gamma} > 0\) is a critical one. Numerical tests show that the components \(c_2, c_4\) seem to have an asymptotically constant behavior,

\[
|c_2(\tau) - c_2(0)| \ll 1, \quad |c_4(\tau) - c_4(0)| \ll 1, \quad \tau \geq 0,
\]

hence, the condition \(c_{2,4}(\infty, x_2, t) = 0\) can only be satisfied if the correction vector \(c\) vanishes identically. Consequently, also in this case, the continuity of the physical variables should be required at \(\Gamma\).
Comments and extensions

- Our approach gives a method to establish transmission conditions for 2D coupled inviscid/viscous compressible flows in a rigorous manner. This approach can be extended to a general smooth interface \( \Gamma \), since the only local parameters involved in our discussion are the normal component of velocity and the corresponding Mach number.
- Also, our approach provides an approximation of the solution of \( (P_2) \), which is a good candidate for the approximation of the real physical flow, by the solution of \( (P_0) \) plus the transition layer correction (if any). This is a consequence of the asymptotic expansion (4.22).

Extending the techniques of [36] we investigate in [12] higher order asymptotic expansions for the solution of \( (P_2) \). This seems rather important because we can get a better description of the flow by retaining more terms of the asymptotic expansion. Let us emphasize that the “regular equations” of order \( k \geq 1 \) are linear partial differential equations. Also, we have succeeded so far to determine the first order “transition layer equations”. We are sure that this asymptotic analysis will reveal further aspects of interface conditions.
- We hope that our reduced model \( (P_0) \) will produce acceptably accurate numerical results, comparable with those obtained by using other methods, such as the \( \chi \)-method in [1, 4]. This is a serious task for future work.
- Our rapid variable works very well wherever \( v_{1,1} \neq 0 \). Its choice is in agreement with the general strategy developed in [30]. Also, we may consider the stationary Navier-Stokes system as to be an elliptic system with respect to \( (x_1, x_2) \) containing a first order convective term for which the technique of [29], p. 300, is available. Moreover, our work in [15–17] on 1D coupled problems supports this point of view. Of course, the present 2D case is more delicate. There is a critical case, \( v_{1,1} = 0 \), when the first derivatives are not present anymore and, hence, the character of the problem changes. But here we suppose that this critical case appears only at some isolated points on \( \Gamma \).
- According to the relation (4.5) we may think that it would be better to take \( \mu = \varepsilon \) and \( k = \alpha \varepsilon \) in \( \Omega^- \), with \( \alpha = c_p/Pr \).
However, then the above theory does not change and we obtain the same results as for \( \alpha = 1 \). On the other hand, in particular physical situations, the thermal conductivity \( k \) is much larger than the viscosity coefficient \( \mu \). In [13] we consider coupled models corresponding to conductive/nonconductive inviscid flows. In this case, we are able to derive the transmission conditions by analytic arguments only. These conditions involve the normal component of the velocity, the corresponding Mach number as well as Poisson’s adiabatic constant \( \gamma \). For the outer boundary conditions, we should require “inviscid” conditions for \( v \) on the whole boundary \( \partial \Omega \), and also some conditions for the temperature on the outer boundary of the thermally conductive subregion.

5 Coupled models for the MPD thruster flow

On the basis of the preceding analysis, we are now going to construct a well-posed formulation for the MPD thruster flow problem.

We choose as an artificial internal boundary a circular segment \( \Gamma \) (see Fig. 8), given in parametrized form

\[
\begin{align*}
  r_\Gamma &= R_0 \sin \theta, \\
  z_\Gamma &= R_0 \cos \theta, \\
  R_0 &> 0, \quad 0 \leq \theta \leq \theta_{\text{max}},
\end{align*}
\]

separating the different model zones with the near field \( \Omega_1 \) containing the MPD thruster and the far field \( \Omega_2 \) corresponding to the tank configuration.

In our further consideration we employ in the near field \( \Omega_1 \) the full MPD system of eqs. (3.1), (3.14)–(3.19), (3.30), (3.34), which consists of 12 PDEs for the vector-valued unknown

\[
\begin{align*}
  \mathbf{u} &= (n_0, n_1, \ldots, n_6, v_r, v_z, E_h, E_e, B)\top, \\
  \mathbf{u} &= \mathbf{u}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1)
\end{align*}
\]

as described in Sect. 3.

![Fig. 8 Internal boundary layer in the MPD thruster/tank configuration plasma flow domain.](image-url)
In the far-field region $\Omega_2$, the complete system will be replaced by simplified mathematical models, based on the physical properties of the plasma flow. Our basic assumption is that the terms expressing the viscosity and the heat conductivity in the extended Navier-Stokes equations are strongly dominated by the convective terms in $\Omega_2$, and that also the (azimuthal) magnetic field $B$ is negligible there. By setting the viscosity $\mu_h$ and the thermal conductivity $k_h$ of the heavy particle flow to zero, and assuming also $B = 0$ in $\Omega_2$ we construct a reduced coupled problem, say $(P_0)$, whose numerical treatment should imply lower computational costs.

In order to formulate the reduced coupled problem rigorously, we first give $(P_0)$ an interpretation based on arguments from the singular perturbation theory. By introducing an artificial viscosity and heat conductivity coefficient $0 < \varepsilon \ll 1$ in $\Omega_2$, we understand $(P_0)$ as a reduced model of a perturbed one, say $(P_\varepsilon)$, obtained by setting

$$
\mu_h = \varepsilon, \quad k_h = \varepsilon, \quad B = 0
$$

in the MPD-system associated with $\Omega_2$. Let us denote by

$$
\mathbf{u}_\varepsilon(r, z, t) := \left( \mathbf{u}_{\Omega_1, \varepsilon}(r, z, t), \mathbf{u}_{\Omega_2, \varepsilon}(r, z, t) \right) \in (\Omega_1, \Omega_2) \times (0, t_1) \to \mathbb{R}^{12} \times \mathbb{R}^{11}
$$

the coupled solution of the regularized problem $(P_\varepsilon)$, which we describe in what follows. Note that, while $\mathbf{u}_{\Omega_1, \varepsilon}$ contains the whole set of 12 flow variables listed in (5.1), the solution vector $\mathbf{u}_{\Omega_2, \varepsilon}$ does not contain $B$, which is supposed to vanish in the far field.

**The coupled problem $(P_\varepsilon)$**

We begin the description of this problem by introducing the partial differential equations involved. As continuity equations, $(P_\varepsilon)$ contains in both subregions of $\Omega$ the conservation equations (3.1) for the densities $n_i$, the species $s_i$, $i = 0, \ldots, 6$.

Furthermore, we consider in $\Omega_2$ the “regularized” momentum equations

$$
\frac{\partial}{\partial t} (\rho \mathbf{v}_r) + \text{div} \mathbf{f}^\varepsilon_{r,\text{invisc}} + \text{div} \mathbf{f}^\varepsilon_{r,\text{visc}} = q^\varepsilon_r, \\
\frac{\partial}{\partial t} (\rho v_z) + \text{div} \mathbf{f}^\varepsilon_{z,\text{invisc}} + \text{div} \mathbf{f}^\varepsilon_{z,\text{visc}} = 0,
$$

with the flux functions

$$
\mathbf{f}^\varepsilon_{r,\text{invisc}} = \left( p_h + p_e + \rho v_r^2 \right) \mathbf{e}_r + (\rho v_r v_z) \mathbf{e}_z, \quad \mathbf{f}^\varepsilon_{r,\text{visc}} = -\tau^\varepsilon_{rr} \mathbf{e}_r - \tau^\varepsilon_{rz} \mathbf{e}_z, \\
\mathbf{f}^\varepsilon_{z,\text{invisc}} = (\rho v_r v_z) \mathbf{e}_r + \left( p_h + p_e + \rho v_z^2 \right) \mathbf{e}_z, \quad \mathbf{f}^\varepsilon_{z,\text{visc}} = -\tau^\varepsilon_{rz} \mathbf{e}_r - \tau^\varepsilon_{zz} \mathbf{e}_z,
$$

containing the “artificial” viscosity terms, obtained from (3.17) by setting $\mu_h = \varepsilon$:

$$
\tau^\varepsilon_{rr} = \frac{\varepsilon}{2} \left( 2 \frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{\partial v_z}{\partial z} \right), \quad \tau^\varepsilon_{rz} = \varepsilon \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right), \quad \tau^\varepsilon_{zz} = \frac{\varepsilon}{2} \left( 2 \frac{\partial v_z}{\partial z} - \frac{v_r}{r} - \frac{\partial v_r}{\partial r} \right)
$$

and the modified source term, obtained from (3.18) by setting $k_h = \varepsilon$ and $B = 0$:

$$
q^\varepsilon_r = \frac{1}{r} \left( p_h + p_e - 2 \varepsilon \left( \frac{v_r}{r} - \frac{1}{3} \text{div} \mathbf{v} \right) \right) \quad \text{in} \quad \Omega_2.
$$

In addition, we consider in $\Omega_2$ the modified energy equation for the heavy particles

$$
\frac{\partial E_h}{\partial t} + \text{div} \mathbf{f}^\varepsilon_{h,\text{invisc}} + \text{div} \mathbf{f}^\varepsilon_{h,\text{visc}} = q^\varepsilon_h,
$$

with the fluxes derived from (3.20)

$$
\mathbf{f}^\varepsilon_{h,\text{invisc}} = (E_h + p_h + p_e) \mathbf{v}, \quad \mathbf{f}^\varepsilon_{h,\text{visc}} = -\left[ \tau^\varepsilon_{rr} v_r + \tau^\varepsilon_{rz} v_z + \varepsilon \frac{\partial T_h}{\partial r} \right] \mathbf{e}_r - \left[ \tau^\varepsilon_{rz} v_r + \tau^\varepsilon_{zz} v_z + \varepsilon \frac{\partial T_h}{\partial z} \right] \mathbf{e}_z,
$$

and the simplified source term, obtained from (3.21) by setting $B = 0$:

$$
q^\varepsilon_h = p_e \text{div} \mathbf{v} + n_e (T_e - T_h) \sum_{i=0}^{6} n_i a_{e i}.
$$
Finally, with the assumption \( j = \text{rot} \mathbf{B}/\mu_0 = 0 \), we deduce from (3.30) the **energy conservation for the electrons** in the simplified form

\[
\frac{\partial E_e}{\partial t} = - \text{div}(E_e \mathbf{v}) - p_e \text{div} \mathbf{v} - \text{div} \left( \frac{3}{2} k T_e \sum_{i=1}^{6} \mathbf{j}_{\text{D},i} \right) + \text{div}(k_e \nabla T_e) + n_e (T_h - T_e) \sum_{i=0}^{6} \alpha_{ei} - \sum_{i=0}^{5} \omega_{i+1} \chi_{i-i+1}. \tag{5.12}
\]

Note that, while the heat conductivity corresponding to the heavy particles \( k_h \) has been set to \( \varepsilon \), at this point we do not make any assumption related to the heat conductivity \( k_e \) corresponding to the electron flow. When performing our asymptotic analysis, we will discuss two possible cases related to the order of magnitude of this quantity: \( k_e \geq k_0 > 0 \) and \( k_e = \mathcal{O}(\varepsilon) \), respectively.

We endow the coupled problem \( (P_2) \) with initial conditions

\[
u_{\Omega_1,e}(r, z, 0) = \nu_{\Omega_1,visc}(r, z), \quad (r, z) \in \Omega_t, \quad i = 1, 2, \tag{5.13}
\]

and with boundary conditions specific to the Navier-Stokes equations as in (4.19). Similar to (4.20) and (4.21), we impose for the coupling at the interface \( \Gamma \) the **natural transmission conditions** consisting of the continuity relations

\[
u_{\Omega_1,e} = \nu_{\Omega_2,e} \text{ on } \Gamma \times (0, t_1), \tag{5.14}
\]

and the continuity of the normal fluxes:

a) For the mass:

\[
\left[ n_i \mathbf{v} + \mathbf{j}_{D,i} \right] (\nu_{\Omega_1,e}) \cdot \mathbf{n} = \left[ n_i \mathbf{v} + \mathbf{j}_{D,i} \right] (\nu_{\Omega_2,e}) \cdot \mathbf{n} \text{ on } \Gamma \times (0, t_1), \quad i = 0, \ldots, 6. \tag{5.15}
\]

b) For the momentum:

\[
\left[ f_{r, \text{invisc}} + f_{r, \text{visc}} \right] (\nu_{\Omega_1,e}) \cdot \mathbf{n} = \left[ f_{r, \text{invisc}} + f_{r, \text{visc}} \right] (\nu_{\Omega_2,e}) \cdot \mathbf{n}, \tag{5.16}
\]

\[
\left[ f_{z, \text{invisc}} + f_{z, \text{visc}} \right] (\nu_{\Omega_1,e}) \cdot \mathbf{n} = \left[ f_{z, \text{invisc}} + f_{z, \text{visc}} \right] (\nu_{\Omega_2,e}) \cdot \mathbf{n}. \tag{5.17}
\]

c) For the heavy-particle energy:

\[
\left[ f_{h, \text{invisc}} + f_{h, \text{visc}} \right] (\nu_{\Omega_1,e}) \cdot \mathbf{n} = \left[ f_{h, \text{invisc}} + f_{h, \text{visc}} \right] (\nu_{\Omega_2,e}) \cdot \mathbf{n}. \tag{5.18}
\]

d) For the electron energy:

\[
[E_e \mathbf{v} + p_e \mathbf{v} + \frac{3}{2} k T_{e,\text{invisc}} \mathbf{j}_{\text{D},e} - k_e \nabla T_e] (\nu_{\Omega_1,e}) \cdot \mathbf{n} = [E_e \mathbf{v} + p_e \mathbf{v} + \frac{3}{2} k T_{e,\text{visc}} \mathbf{j}_{\text{D},e} - k_e \nabla T_e] (\nu_{\Omega_2,e}) \cdot \mathbf{n}. \tag{5.19}
\]

As in the simpler case of a 2D Cartesian one-phase fluid model, it is expected that no boundary layer appears at the “viscous” side of the interface \( \Gamma \), contained in \( \Omega_1 \), and, consequently, we shall investigate only the possible boundary layer associated with the right (“inviscid”) side of \( \Gamma \).

In order to do this, we first have to establish the **rapid variable**. Hereby, we use intuitive arguments. By setting \( \mu_h = k_h = \varepsilon \) in \( \Omega_2 \), the corresponding system of equations for \( (P_2) \) is parabolic with spatially elliptic operators with respect to \((r, z)\); it also includes first order spatial derivatives, due to the convective part. So, according to [28, p. 300], our rapid variable can be defined as

\[
\tau = \frac{1}{\varepsilon} \left( \sqrt{r^2 + z^2} - R_0 \right) \text{ for } (r, z) \in \Omega_2. \tag{5.20}
\]

We will see that this choice works well and takes into account the principal structure of the model (see [30]). We will be able to transfer our local analysis of Sect. 4 to the case of the curved interface \( \Gamma \), if the \( x_1 \)-coordinate in (4.27) is now replaced by the direction normal to \( \Gamma \) and the normal velocity \( v_1 \) becomes \( \mathbf{v} \cdot \mathbf{n} \). In the transition layer along the circular interface, the original coordinates \((r, z)\) are replaced by the new coordinates \((\theta, \tau)\):

\[
r = (R_0 + \varepsilon \tau) \sin \theta, \quad z = (R_0 + \varepsilon \tau) \cos \theta, \tag{5.21}
\]

or, equivalently,

\[
\theta = \arctan \frac{r}{z}, \quad \tau = \frac{1}{\varepsilon} \left( \sqrt{r^2 + z^2} - R_0 \right). \tag{5.22}
\]
Then the partial derivatives transform as
\[
\frac{\partial}{\partial r} = \frac{\cos \theta}{R_0 + \varepsilon} \cdot \frac{\partial}{\partial \theta} + \frac{\sin \theta}{\varepsilon} \cdot \frac{\partial}{\partial r} \quad \text{and} \quad \frac{\partial}{\partial z} = -\frac{\sin \theta}{R_0 + \varepsilon} \cdot \frac{\partial}{\partial \theta} + \frac{\cos \theta}{\varepsilon} \cdot \frac{\partial}{\partial r}.
\]
Consequently, we shall use the following expansion formulas in the transition layer:
\[
\begin{align*}
\frac{\partial}{\partial r} &= \frac{\cos \theta}{R_0} \cdot \frac{\partial}{\partial \theta} + \frac{\sin \theta}{\varepsilon} \cdot \frac{\partial}{\partial r} + O(\varepsilon), \\
\frac{\partial}{\partial z} &= -\frac{\sin \theta}{R_0} \cdot \frac{\partial}{\partial \theta} + \frac{\cos \theta}{\varepsilon} \cdot \frac{\partial}{\partial r} + O(\varepsilon), \\
\frac{\partial^2}{\partial r^2} &= \frac{\sin^2 \theta}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{\cos \theta}{\varepsilon R_0} \left[ 2 \sin \theta \frac{\partial^2}{\partial \tau \partial \theta} + \cos \theta \frac{\partial}{\partial r} \right] + \frac{\cos^2 \theta}{R_0^2} \frac{\partial^2}{\partial \theta^2} - \frac{2 \sin \theta}{2R_0^2} \frac{\partial}{\partial r} + O(\varepsilon), \\
\frac{\partial^2}{\partial z^2} &= \frac{\cos^2 \theta}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{\sin \theta}{\varepsilon R_0} \left[ -2 \cos \theta \frac{\partial^2}{\partial \tau \partial \theta} + \sin \theta \frac{\partial}{\partial r} \right] + \frac{\sin^2 \theta}{R_0^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin 2 \theta}{2R_0^2} \frac{\partial}{\partial r} + O(\varepsilon).
\end{align*}
\]

The degenerate coupled problem \((P_0)\)

We first note that for \(\varepsilon \to 0\), the partial differential equations of the degenerate coupled problem represent the coupling of the extended Navier-Stokes equations in \(\Omega_1\) with the extended Euler equations in \(\Omega_2\).

It can also be shown that the initial conditions for \((P_0)\) are the same as (5.13) for \((P_e)\), and that the boundary conditions at \(\partial \Omega_1 \setminus \Gamma\) are of Navier-Stokes type, whereas the boundary conditions at \(\partial \Omega_2 \setminus \Gamma\) change to conditions which are specific to inviscid flows. In what follows, we develop transmission conditions for \((P_0)\) across the artificial interface \(\Gamma\).

1. Euler \(\to\) Navier-Stokes transmission conditions

By using the results concerning the Navier-Stokes/Euler coupling for one-phase fluid flows mentioned in Sect. 4 (for a complete reading, see [111]), we employ the conservation of normal fluxes across \(\Gamma\) as transmission conditions for the second-order system of the extended Navier-Stokes equations in \(\Omega_1\).

Our aim is to formulate these transmission conditions in such a way, that the contributions of all possible terms are taken into account in the calculation of the normal fluxes.

Note that the unit normal to \(\Gamma\) at some point \((r_\Gamma, z_\Gamma) = (R_0 \sin \theta, R_0 \cos \theta)\) is \(n = \sin \theta \cdot e_r + \cos \theta \cdot e_z\). For the ease of reading, we will denote by \([\cdot]_{\text{left}}\) and \([\cdot]_{\text{right}}\) the limit values of different flow quantities on the interface, when taken from the near field \(\Omega_1\) and from the far field \(\Omega_2\), respectively.

From the conservation of mass equations (3.1) we first have the following flux conditions
\[
[n_i \nu + j_{D,i}]_{\text{left}} \cdot n = [n_i \nu + j_{D,i}]_{\text{right}} \cdot n \quad \text{on} \quad \Gamma \times (0, t_1) \quad \text{for all} \quad i = 0, \ldots, 6.
\]
(5.24)

Corresponding to the momentum equation in the radial direction (3.14) we derive the flux condition
\[
[f_r, \text{invisc} + f_r, \text{visc} - \frac{2}{3} \mu_h \nu]_{\text{left}} \cdot n = [f_r, \text{invisc}]_{\text{right}} \cdot n \quad \text{on} \quad \Gamma \times (0, t_1).
\]
(5.25)

The term \(-\frac{2}{3} \mu_h \nu\) in (5.25) originates from the source term \(q_r\) defined by (3.18), and is introduced here for the first time; for \(q_r\), we assume that \(\mu_h\) is continuous in some vicinity of \(\Gamma\) included in \(\overline{\Omega}_1\) and, therefore, \(\mu_h\) is considered to be locally constant. As already mentioned, we set \(\mu_h = k_h = 0\) in \(\Omega_2\). In addition, the assumption \(B = 0\) implies \(j = 0\) and \(\nu = \nu_e\).

Now, the flux condition corresponding to the longitudinal momentum equation (3.15) is
\[
[f_z, \text{invisc} + f_z, \text{visc}]_{\text{left}} \cdot n = [f_z, \text{invisc}]_{\text{right}} \cdot n \quad \text{on} \quad \Gamma \times (0, t_1).
\]
(5.26)

From (3.19) we derive the flux conditions for \(E_h\) by assuming \(B = 0\) on \(\Gamma\) and incorporating \(q_h\) from (3.21) in \(\overline{\Omega}_1\), where \(p_e\) is assumed to be locally constant at \(\Gamma\):
\[
[f_h, \text{invisc}(\text{with} \ B = 0) + f_h, \text{visc} - p_e \nu]_{\text{left}} \cdot n = [(E_h + p_h + p_e) \nu - p_e \nu]_{\text{right}} \cdot n \quad \text{on} \quad \Gamma \times (0, t_1),
\]
which implies
\[
[(E_h + p_h) \nu + f_h, \text{visc}]_{\text{right}} \cdot n = [(E_h + p_h) \nu]_{\text{right}} \cdot n \quad \text{on} \quad \Gamma \times (0, t_1).
\]
(5.27)
Finally, from eq. (3.30) we derive the flux condition for $E_e$, i.e.,

$$
[E_e \mathbf{v} + p_e \mathbf{v} + \frac{1}{2} \mathbf{k} T_e \mathbf{j}_{D,e} - k_e \nabla T_e]_{\text{left}} \cdot \mathbf{n} = [E_e \mathbf{v} + p_e \mathbf{v} + \frac{1}{2} \mathbf{k} T_e \mathbf{j}_{D,e} - k_e \nabla T_e]_{\text{right}} \cdot \mathbf{n}
$$

(5.28)
on $\Gamma \times (0, t_1)$. The relations (5.24)–(5.28) contain 11 necessary flux conditions at $\Gamma$ for $(P_0)$. Note that every term appearing under the divergence operator “$\nabla$” in the MPD system is included in the flux conditions (5.24)–(5.28).

All these transmission conditions for the extended Navier-Stokes equations are here derived for the first time in terms of formal asymptotics.

2. **Navier-Stokes → Euler transmission conditions**

In addition to the flux conditions (5.24)–(5.28), we now need for the first-order hyperbolic extended Euler system transmission conditions in form of continuity relations for the flow variables. These conditions can be obtained by applying asymptotic analysis. Since a possible boundary layer is expected only on the right side of $\Gamma$, included in $\Omega_2$ (see Fig. 8), we consider the zeroth order expansions (similar to (4.22)) in the form

$$
u_{\Omega_1,e}(r, z, t) = \nu_{\Omega_1}(r, z, t) + \text{rem}_{\Omega_1,e}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
u_{\Omega_2,e}(r, z, t) = \nu_{\Omega_2}(r, z, t) + \mathbf{c}(\tau, \theta, t) + \text{rem}_{\Omega_2,e}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1).
$$

(5.29)

(5.30)

By $\text{rem}$ we denote the remainder terms. Nevertheless, the reduced (degenerate) system of partial differential equations is satisfied by the zeroth order terms of the regular series

$$
\mathbf{u} := (\nu_{\Omega_1}, \nu_{\Omega_2}) : (\Omega_1, \Omega_2) \times (0, t_1) \rightarrow \mathbb{R}^{12} \times \mathbb{R}^{11},
$$

and is supplemented by the flux conditions (5.24)–(5.28) as well as by the corresponding continuity conditions. Because the magnetic field is assumed to be totally absent in $\Omega_2$, the representation (5.30) contains 11 components only. In particular, the vector-valued internal layer function

$$
\mathbf{c}(\tau, \theta, t) = (c_{n_0}, c_{n_1}, \ldots, c_{n_9}, c_{v_1}, c_{v_2}, c_{E_h}, c_{E_e})^T
$$

(5.31)
collects the internal layer corrections of order zero corresponding to the flow variables in $\Omega_2$. Thus, (5.29)–(5.30) can be written component-wise for the densities $n_i$ of the species:

$$
n_{i,\Omega_1,e}(r, z, t) = n_{i,\Omega_1}(r, z, t) + \text{rem}_{n_{i,\Omega_1,e}}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
n_{i,\Omega_2,e}(r, z, t) = n_{i,\Omega_2}(r, z, t) + c_{n_i}(\tau, \theta, t) + \text{rem}_{n_{i,\Omega_2,e}}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1),
$$

(5.32)

(5.33)

for $i = 0, \ldots, 6$), for the velocity components:

$$
v_{r,\Omega_1,e}(r, z, t) = v_{r,\Omega_1}(r, z, t) + \text{rem}_{v_{r,\Omega_1,e}}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
v_{r,\Omega_2,e}(r, z, t) = v_{r,\Omega_2}(r, z, t) + c_{v_r}(\tau, \theta, t) + \text{rem}_{v_{r,\Omega_2,e}}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1),
$$

$$
v_{z,\Omega_1,e}(r, z, t) = v_{z,\Omega_1}(r, z, t) + \text{rem}_{v_{z,\Omega_1,e}}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
v_{z,\Omega_2,e}(r, z, t) = v_{z,\Omega_2}(r, z, t) + c_{v_z}(\tau, \theta, t) + \text{rem}_{v_{z,\Omega_2,e}}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1),
$$

(5.34)

(5.35)

(5.36)

(5.37)

for the heavy-particle energy:

$$
E_{h,\Omega_1,e}(r, z, t) = E_{h,\Omega_1}(r, z, t) + \text{rem}_{E_{h,\Omega_1,e}}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
E_{h,\Omega_2,e}(r, z, t) = E_{h,\Omega_2}(r, z, t) + c_{E_h}(\tau, \theta, t) + \text{rem}_{E_{h,\Omega_2,e}}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1),
$$

(5.38)

(5.39)

and, finally, for the electron energy:

$$
E_{e,\Omega_1,e}(r, z, t) = E_{e,\Omega_1}(r, z, t) + \text{rem}_{E_{e,\Omega_1,e}}(r, z, t), \quad (r, z, t) \in \Omega_1 \times (0, t_1),
$$

$$
E_{e,\Omega_2,e}(r, z, t) = E_{e,\Omega_2}(r, z, t) + c_{E_e}(\tau, \theta, t) + \text{rem}_{E_{e,\Omega_2,e}}(r, z, t), \quad (r, z, t) \in \Omega_2 \times (0, t_1).
$$

(5.40)

(5.41)

In order to obtain the desired continuity conditions for the coupled problem $(P_0)$, we need to formulate and to solve a correction problem.
The problem \((P_c)\) for the corrections \(c(\cdot, \theta, t) : \mathbb{R}_+ \to \mathbb{R}^1\)

In a similar manner as for the problem treated in Sect. 4, the problem for the internal layer corrections consists of nonlinear ordinary differential equations for \(c\). As the present system is rather complex, we shall try to simplify the computation by employing some additional, new arguments.

For instance, the only part of the expansion (5.33) which contributes to the derivation of the ordinary differential equations for \((P_c)\), is \(n_i(R_0 \sin \theta, R_0 \cos \theta, t) + c_n(\tau, \theta, t)\). (Here, we remove the subscript \(\Omega_2\).) Following the traditional technique, we suppose that the variables

\[(r, z) = ((R_0 + \varepsilon \tau) \sin \theta, (R_0 + \varepsilon \tau) \cos \theta)\]

are frozen at \((R_0 \sin \theta, R_0 \cos \theta)\).

Correspondingly, we formulate the ODE’s describing the transition layer correction functions incorporated in \(c(\cdot, \theta, t)\):

### A. Transition layer equations corresponding to the mass conservation

If we insert the expansions (5.33) into the differential equations (3.1) for the densities in \(\Omega_2\), then we can see that the transition layer equations are generated by the diffusion terms “\(\nabla \cdot \)”. We have

\[
\mathbf{j}_{D,i} = - \left( \sum_{l=0}^{6} n_l \right) \nabla \left[ \frac{n_i}{\sum_{l=0}^{6} (1 + l) n_l} \right] + n_i \sum_{k=0}^{6} D_{km} \nabla \left[ \frac{n_k}{\sum_{l=0}^{6} (1 + l) n_l} \right] =: C_{i,r} e_r + C_{i,z} e_z, \tag{5.42}
\]

with

\[
C_{i,r}(r, z, t) := - \left( \sum_{l=0}^{6} n_l \right) D_{im} \frac{\partial d_i}{\partial r} + n_i \sum_{k=0}^{6} D_{km} \frac{\partial d_k}{\partial r}, \tag{5.43}
\]

and

\[
C_{i,z}(r, z, t) := - \left( \sum_{l=0}^{6} n_l \right) D_{im} \frac{\partial d_i}{\partial z} + n_i \sum_{k=0}^{6} D_{km} \frac{\partial d_k}{\partial z}, \tag{5.44}
\]

where we denote

\[
d_i = \frac{n_i + c_n i}{\sum_{l=0}^{6} (1 + l) (n_l + c_n i)} \quad \text{for} \quad i = 0, \ldots, 6. \tag{5.45}
\]

The divergence of the diffusion term \(\mathbf{j}_{D,i}\) has the form

\[
\mathbf{div} \mathbf{j}_{D,i} = \frac{1}{r} \frac{\partial}{\partial r} \left( r C_{i,r} \right) + \frac{\partial}{\partial z} C_{i,z} = \frac{C_{i,r}}{r} + \frac{\partial C_{i,r}}{\partial r} + \frac{\partial C_{i,z}}{\partial z}. \tag{5.46}
\]

The representations (5.43) and (5.44), together with the differentiation formulae from (5.23) imply

\[
\frac{\partial C_{i,r}}{\partial r} = \frac{\sin \theta}{\varepsilon} \frac{\partial C_{i,r}}{\partial r} + \frac{\cos \theta}{R_0} \frac{\partial C_{i,r}}{\partial \theta} + O(\varepsilon) \tag{5.47}
\]

\[
= \frac{\sin \theta}{\varepsilon} \frac{\partial}{\partial r} \left[ - \sum_{l=0}^{6} (n_l + c_n i) D_{im} \left( \frac{\sin \theta}{\varepsilon} \frac{\partial d_i}{\partial r} + O(1) \right) + (n_i + c_n i) \sum_{k=0}^{6} D_{km} \left( \frac{\sin \theta}{\varepsilon} \frac{\partial d_k}{\partial r} + O(1) \right) \right],
\]

and

\[
\frac{\partial C_{i,z}}{\partial z} = \frac{\cos \theta}{\varepsilon} \frac{\partial C_{i,z}}{\partial r} - \frac{\sin \theta}{R_0} \frac{\partial C_{i,z}}{\partial \theta} + O(\varepsilon) \tag{5.48}
\]

\[
= \frac{\cos \theta}{\varepsilon} \frac{\partial}{\partial r} \left[ - \sum_{l=0}^{6} (n_l + c_n i) D_{im} \left( \frac{\cos \theta}{\varepsilon} \frac{\partial d_i}{\partial r} + O(1) \right) + (n_i + c_n i) \sum_{k=0}^{6} D_{km} \left( \frac{\cos \theta}{\varepsilon} \frac{\partial d_k}{\partial r} + O(1) \right) \right]
\]

for all \(i = 0, \ldots, 6\). By using (5.47) and (5.48) in the expression (5.46) for the divergence, and by equating the coefficients of \(\varepsilon^{-2}\), we obtain the transition layer equations corresponding to the mass conservation equations (3.1) in form of the identities

\[
\sin \theta \cdot \frac{\partial}{\partial r} \left\{ - \sum_{l=0}^{6} (n_l + c_n i) D_{im} \sin \theta \frac{\partial d_i}{\partial r} + (n_i + c_n i) \sum_{k=0}^{6} D_{km} \sin \theta \frac{\partial d_k}{\partial r} \right\}
\]
Assuming first that

Therefore, these relations yield

The flow variables \( n_i \) are calculated at \((R_0 \sin \theta, R_0 \cos \theta, t)\).

Actually, as the transition layer terms vanish for \( \tau \to \infty \), i.e.

we deduce from (5.49) the relations

Denote

We have \( b > 0 \) and \( b_i > 0 \) (\( i = 0, \ldots, 6 \)), since these are densities. According to (5.51), then the following linear algebraic system in \( A_i \) is satisfied:

Assuming first that \( \sum_{k=0}^{n_i} A_k \neq 0 \), we see that \( A_i \neq 0 \) for \( i = 0, \ldots, 6 \) as well. Hence, six of the coefficients \( A_i \) are independent and the system (5.52) reduces to

Therefore,

In other words,

Hence, the seven corrections \( c_{n_i} (i = 0, \ldots, 6) \) are related by six independent equations. An additional transition layer equation can be derived from the conservation of mass (3.5):

where the correction \( c_\mu \) corresponding to the total density \( \rho \) is defined by

Consequently, relation (5.56) can be written as

with \((n_i, v_r, v_\theta, v_z) := (n_i, v_r, v_\theta) (R_0 \sin \theta, R_0 \cos \theta, t)\).
In the special case when \( \sum_{k=0}^{6} A_k = 0 \) on some \( \tau \)-interval, then also \( A_i = 0 (i = 0, \ldots, 6) \) on that interval (cf. (5.52)). It follows by the definition of \( A_i \) that then the quantities \( d_i \) are independent of \( \tau \), i.e.,
\[
\frac{n_i + c_{n_i}}{\sum_{l=0}^{6}(1 + l) (n_l + c_{n_l})} = \frac{n_i}{\sum_{l=0}^{6}(1 + l)n_l} \quad \text{for } i = 0, \ldots, 6.
\]
Therefore
\[
\frac{n_i + c_{n_i}}{n_0 + c_{n_0}} = \frac{n_i}{n_0},
\]
which implies
\[
c_{n_i}(\tau, \theta, t) = \left( \frac{n_i}{n_0} \right) (R_0 \sin \theta, R_0 \cos \theta, t) \cdot c_{n_0}(\tau, \theta, t) \quad \text{for } i = 1, \ldots, 6. \tag{5.59}
\]
These equations together with (5.58), form again a system of seven equations for the components \( c_{n_i}(\tau, \theta, t), i = 0, \ldots, 6 \) of the internal layer correction \( c \).

### B. Transition layer equations corresponding to the momentum equations

By using the differentiation formulae from (5.23), we get from the momentum equation in the radial direction (5.4) via collecting the coefficients of \( \varepsilon^{-1} \) the transition layer equation
\[
\sin \theta \cdot \frac{\partial}{\partial \tau} \left[ c_{p_h} + c_{p_o} + (\rho + c_p) (v_r + c_{v_r})^2 \right] + \cos \theta \cdot \frac{\partial}{\partial \tau} \left[ (\rho + c_p) (v_r + c_{v_r}) (v_z + c_{v_z}) \right] - \frac{4}{3} \sin^2 \theta \cdot \frac{\partial^2 c_{v_r}}{\partial \tau^2} + \frac{2}{3} \sin \theta \cdot \cos \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} - \sin \theta \cdot \cos \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} - \cos^2 \theta \cdot \frac{\partial^2 c_{v_r}}{\partial \tau^2} = 0,
\]
where, again, the flow variables \( \rho, v_r, \) and \( v_z \) are calculated at \( (R_0 \sin \theta, R_0 \cos \theta, t) \). A first integration with respect to \( \tau \) implies
\[
\sin \theta \cdot \left[ c_{p_h} + c_{p_o} + (\rho + c_p) (v_r + c_{v_r})^2 \right] + \cos \theta \cdot (\rho + c_p) (v_r + c_{v_r}) (v_z + c_{v_z}) - \left( 1 + \frac{1}{3} \sin^2 \theta \right) \cdot \frac{\partial c_{v_r}}{\partial \tau} - \frac{1}{3} \sin \theta \cos \theta \cdot \frac{\partial c_{v_z}}{\partial \tau} = \sin \theta \cdot c_{p_h} + c_{p_o} + \rho c_{v_r} (v_r \sin \theta + v_z \cos \theta) + \sin \theta \cdot (c_{p_h} + c_{p_o}) = 0. \tag{5.60}
\]

Beginning now from the momentum equation in the longitudinal direction (5.5) and applying the same procedure, we obtain the relation
\[
\sin \theta \cdot \frac{\partial}{\partial \tau} \left[ (\rho + c_p) (v_r + c_{v_r}) (v_z + c_{v_z}) \right] + \cos \theta \cdot \frac{\partial}{\partial \tau} \left[ c_{p_h} + c_{p_o} + (\rho + c_p) (v_z + c_{v_z})^2 \right] - \sin^2 \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} - \sin \theta \cdot \cos \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} - \frac{4}{3} \cos^2 \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} + \frac{2}{3} \sin \theta \cdot \cos \theta \cdot \frac{\partial^2 c_{v_z}}{\partial \tau^2} = 0.
\]
Integrating once with respect to \( \tau \) and using the conditions (5.50), we arrive at
\[
(v_z + c_{v_z}) \left[ \sin \theta \cdot (\rho + c_p) (v_r + c_{v_r}) + \cos \theta \cdot (\rho + c_p) (v_z + c_{v_z}) \right] + \cos \theta \cdot (c_{p_h} + c_{p_o}) - \left( 1 + \frac{1}{3} \cos^2 \theta \right) \frac{\partial c_{v_z}}{\partial \tau} - \frac{1}{3} \sin \theta \cos \theta \cdot \frac{\partial c_{v_z}}{\partial \tau} = \rho c_{v_z} (v_r \sin \theta + v_z \cos \theta), \tag{5.62}
\]
where, again, the right-hand side is evaluated at \( (R_0 \sin \theta, R_0 \cos \theta, t) \). Employing (5.56), we can rewrite (5.62) as
\[
- \frac{1}{3} \sin \theta \cos \theta \cdot \frac{\partial c_{v_z}}{\partial \tau} - \left( 1 + \frac{1}{3} \cos^2 \theta \right) \frac{\partial c_{v_z}}{\partial \tau} + \rho c_{v_z} (v_r \sin \theta + v_z \cos \theta) + \cos \theta \cdot (c_{p_h} + c_{p_o}) = 0. \tag{5.63}
\]
The system of eqs. (5.61), (5.63) can uniquely be resolved with respect to \( \partial c_{v_r}/\partial \tau \) and \( \partial c_{v_z}/\partial \tau \). We obtain
\[
\frac{\partial c_{v_r}}{\partial \tau} = -\frac{3}{4} \rho (v_r \sin \theta + v_z \cos \theta) \left[ \frac{1}{3} \sin \theta \cos \theta c_{v_z} + \left( 1 + \frac{1}{3} \sin^2 \theta \right) c_{v_z} \right] + \frac{3}{4} \sin \theta (c_{p_h} + c_{p_o}), \tag{5.64}
\]
\[
\frac{\partial c_{v_z}}{\partial \tau} = -\frac{3}{4} \rho (v_r \sin \theta + v_z \cos \theta) \left[ \frac{1}{3} \sin \theta \cos \theta c_{v_z} + \left( 1 + \frac{1}{3} \sin^2 \theta \right) c_{v_z} \right] + \frac{3}{4} \cos \theta (c_{p_h} + c_{p_o}). \tag{5.65}
\]
C. Transition layer equations corresponding to the energy equation for the heavy-particle flow

Now, we can deduce from (5.9) the equation

\[
\sin \theta \frac{\partial}{\partial \tau} \left[ \left( E_h + c_{E_h} + p_h + c_{p_h} + p_e + c_{p_e} \right) (v_r + c_{v_r}) \right] \\
+ \cos \theta \frac{\partial}{\partial \tau} \left[ \left( E_h + c_{E_h} + p_h + c_{p_h} + p_e + c_{p_e} \right) (v_z + c_{v_z}) \right] \\
- \sin \theta \frac{\partial}{\partial \tau} \left\{ \frac{2}{3} (v_r + c_{v_r}) \frac{\partial}{\partial \tau} (2 \sin \theta c_{v_r} - \cos \theta c_{v_z}) + (v_z + c_{v_z}) \frac{\partial}{\partial \tau} (\sin \theta c_{v_z} + \cos \theta c_{v_r}) + \sin \theta \frac{\partial c_{T_h}}{\partial \tau} \right\} \\
- \cos \theta \frac{\partial}{\partial \tau} \left\{ (v_r + c_{v_r}) \frac{\partial}{\partial \tau} (\sin \theta c_{v_z} + \cos \theta c_{v_r}) + \frac{2}{3} (v_z + c_{v_z}) \frac{\partial}{\partial \tau} (2 \cos \theta c_{v_r} - \sin \theta c_{v_z}) + \cos \theta \frac{\partial c_{T_h}}{\partial \tau} \right\} \\
= \left( p_e + c_{p_e} \right) \frac{\partial}{\partial \tau} (\sin \theta c_{v_r} + \cos \theta c_{v_z}).
\]

(5.66)

While the convective part of (5.9) leads only to first order derivatives of different correction functions, its dissipative part leads to quadratic terms of the form

\[
\frac{\partial}{\partial \tau} \left( c_{v_r} \frac{\partial c_{v_r}}{\partial \tau} \right), \quad \frac{\partial}{\partial \tau} \left( c_{v_z} \frac{\partial c_{v_z}}{\partial \tau} \right), \quad \frac{\partial}{\partial \tau} \left( c_{v_z} \frac{\partial c_{v_r}}{\partial \tau} \right), \quad \frac{\partial}{\partial \tau} \left( c_{v_r} \frac{\partial c_{v_z}}{\partial \tau} \right), \quad \text{and} \quad \frac{\partial^2 c_{T_h}}{\partial \tau^2}.
\]

Again, the flow variables \( E_h, p_h, p_e, \) and \( v \) are evaluated at \((R_0 \sin \theta, R_0 \cos \theta, t)\).

D. Transition layer equations corresponding to the energy equation for the electron field

In order to derive the transition layer equation corresponding to eq. (3.30), we collect the coefficients of \( \varepsilon^{-2} \) which are generated by

\[-\text{div} \left( \frac{3}{2} k_T \varepsilon \sum_{i=1}^{6} j_{D,i} \right) + \text{div} (k_e \nabla T_e) .\]

Taking into account the definition of \( j_{D,i} \) and the differentiation formulae from (5.23), we obtain the relation

\[
\frac{1}{\varepsilon^2} \frac{\partial}{\partial \tau} \left\{ -\frac{3}{2} k (T_e + c_{T_e}) \sum_{i=1}^{6} i \left[ -\sum_{l=0}^{6} (n_l + c_{n_l}) D_{llm} \frac{\partial d_l}{\partial \tau} \right] \right\} \\
+ \frac{1}{\varepsilon^2} \frac{\partial}{\partial \tau} \left\{ -\frac{3}{2} k (T_e + c_{T_e}) \sum_{i=1}^{6} i (n_i + c_{n_i}) \sum_{k=0}^{6} D_{km} \frac{\partial d_k}{\partial \tau} \right\} + \frac{1}{\varepsilon^2} \frac{\partial^2 c_{T_e}}{\partial \tau^2} + \mathcal{O} \left( \frac{1}{\varepsilon^2} \right) + \mathcal{O}(1) = 0,
\]

hence, requiring the coefficient of \( \varepsilon^{-2} \) to vanish, performing an integration with respect to \( \tau \) and using the conditions at infinity (5.50), we arrive at the equation

\[
-\frac{3}{2} k (T_e + c_{T_e}) \sum_{i=1}^{6} i \left[ -\sum_{l=0}^{6} (n_l + c_{n_l}) D_{llm} \frac{\partial d_l}{\partial \tau} + (n_i + c_{n_i}) \sum_{k=0}^{6} D_{km} \frac{\partial d_k}{\partial \tau} \right] + k_e \frac{\partial c_{T_e}}{\partial \tau} = 0.
\]

(5.67)

Finally, we distinguish two cases:

1. If we assume that the electron fluid is still thermally conductive in \( \Omega_2 \), i.e., the heat conductivity \( k_e \) satisfies a relation of the form

\[ k_e \geq k_0 > 0, \]

then we can use the identity (5.51) and get from (5.67) the relation

\[ \frac{\partial c_{T_e}}{\partial \tau} = 0 \quad \text{which implies} \quad c_{T_e} \equiv 0. \]

This property yields necessarily the continuity condition for the electron temperature:

\[ T_{e,\text{left}} = T_{e,\text{right}} \quad \text{on} \quad \Gamma \times (0, t_1). \]

(5.68)

This is now a first continuity condition to be incorporated into the formulation of the coupled degenerate problem (5.50).
2. If we assume that also \( k_e \) becomes “very small” in \( \Omega_2 \), as e.g. to be of the same order of magnitude as \( k_h \), then it is natural to suppose in the formulation of \((P_2)\) that \( k_e(r, z, t) = a_1(r, z, t) \cdot \varepsilon \) with \( 0 < a_1(r, z, t) = O(1) \) in \( \Omega_2 \). Then, the corresponding transition layer equation reads after one integration with respect to \( \tau \):

\[
\begin{align*}
\frac{\partial r^2}{\partial \tau} &= \sin \theta \cdot (E_e + c_{E_h}) (v_r + c_{v_r}) + \cos \theta \cdot (E_e + c_{E_h}) (v_z + c_{v_z}) \\
&\quad - (p_e + c_{p_e}) (\sin \theta \cdot c_{v_r} + \cos \theta \cdot c_{v_z}) - \sin \theta \cdot E_e v_r - \cos \theta \cdot E_e v_z.
\end{align*}
\]

(5.69)

Of course, in this case, in the reduced model \((P_0)\) we set \( k_e = 0 \) in \( \Omega_2 \).

Other situations concerning the thermal conductivity of the two components of the fluid can also be discussed. However, it seems that the most appropriate physical assumption is that the electron fluid is still thermally conductive, as discussed before.

The nonlinear ODE system for the vector-valued correction functions \( c \) contains the eqs. (5.55) (or (5.59)), together with (5.58) for the mass conservation, the eqs. (5.64) and (5.65) for the momentum, the eq. (5.66) for the heavy-particle energy, and, finally, the relation (5.67) (or (5.69)) for the electron energy.

We point out that in all the above transition layer equations the symbols \( n_i, v_r, v_z, \ldots \) represent the values of \( n_i, \Omega_2, v_r, \Omega_2, v_z, \Omega_2, \ldots \) at \((R_0 \sin \theta, R_0 \cos \theta, t)\).

If we choose the correction \( c_{n_i} \) for the species densities \((i = 0, \ldots, 6)\), \( c_{v_r} \) and \( c_{v_z} \) for the velocities, \( c_{T_h} \) and \( c_{T_e} \) for the temperatures as basic boundary layer corrections, then \( c_{p_e}, c_{p_h}, c_{E_h}, c_{E_e} \) can be defined by means of the state equations, i.e.,

\[
\begin{align*}
\frac{\partial n_0}{\partial \tau} &= \frac{k}{m_h} \left[ (\rho + c_p) (T_h + c_{T_h}) - \rho T_h \right], \\
\frac{\partial n_i}{\partial \tau} &= \frac{k}{m_h} \left[ \sum_{i=1}^{6} n_i + c_{n_i} - T_e \sum_{i=1}^{6} i n_i \right], \\
\frac{\partial n_6}{\partial \tau} &= \frac{k}{m_h} \left[ (\rho + c_p) (T_h + c_{T_h}) + \frac{1}{2} (\rho + c_p) (v_r + c_{v_r})^2 + (v_z + c_{v_z})^2 - E_h \right], \\
\frac{\partial E_{\text{e}}}{\partial \tau} &= \frac{k}{m_h} \left[ \sum_{i=1}^{6} i n_i + c_{E_{\text{e}}} - E_{\text{e}} \right],
\end{align*}
\]

with \((\rho, T_h, T_e, n_i, v_r, v_z, E_h, E_e)\) denoting the values of \((\rho, T_h, T_e, n_i, v_r, v_z, E_h, E_e)_{\Omega_2}\) at \((R_0 \sin \theta, R_0 \cos \theta, t)\).

Now, the equations of the problem \((P_2)\) are defined. The corrections take care of possible discontinuities of the solution of the reduced problem \((P_0)\) at \( \Gamma \). These jump discontinuities define the initial conditions for the corrections. More precisely, we have

\[
\begin{pmatrix}
  c_{n_0} \\
  \vdots \\
  c_{n_6} \\
  c_{v_r} \\
  c_{v_z} \\
  c_{T_h} \\
  c_{T_e}
\end{pmatrix}
\begin{pmatrix}
  0, \theta, t
\end{pmatrix}
=
\begin{pmatrix}
  (n_0) \\
  \vdots \\
  (n_6) \\
  v_r \\
  v_z \\
  T_h \\
  T_e
\end{pmatrix}
\begin{pmatrix}
  0, \theta, t
\end{pmatrix}
\begin{pmatrix}
  \left(\rho, p_h, E_e, E_h\right)_\text{left} \\
  \left(\rho, p_h, E_e, E_h\right)_\text{right}
\end{pmatrix}
\begin{pmatrix}
  (R_0 \sin \theta, R_0 \cos \theta, t)
\end{pmatrix}.
\]

(5.70)

Then the additional jumps

\[
\begin{pmatrix}
  c_{p_e} \\
  c_{p_h} \\
  c_{p_v} \\
  c_{E_e} \\
  c_{E_h}
\end{pmatrix}
\begin{pmatrix}
  0, \theta, t
\end{pmatrix}
=
\begin{pmatrix}
  (\rho) \\
  (p_h) \\
  (p_e) \\
  (E_h) \\
  (E_e)
\end{pmatrix}
\begin{pmatrix}
  0, \theta, t
\end{pmatrix}
\begin{pmatrix}
  \left(\rho, p_h, E_e, E_h\right)_\text{left} \\
  \left(\rho, p_h, E_e, E_h\right)_\text{right}
\end{pmatrix}
\begin{pmatrix}
  (R_0 \sin \theta, R_0 \cos \theta, t)
\end{pmatrix}.
\]

(5.71)

These jump discontinuities can be obtained from the quantities in (5.70) a posteriori.

Now we are in the position to apply the ideas presented in Sect. 4. We consider the correction functions as flags, indicating at any point situated on the interface, and at any time, whether and perhaps which continuity conditions have to be employed at the coupling boundary. More precisely, if some of the corrections are identical zero, then we get from (5.70) and (5.71) the corresponding continuity conditions at the coupling boundary \( \Gamma \). For example, under the assumption \( k_e \geq k_0 > 0 \) we have obtained the continuity condition (5.68).
In order to formulate the reduced model \( (P_0) \) completely, we should require initial conditions for all the flow variables and, in addition, specific outer boundary conditions. More specifically, let us first assume that there is no outlet on \( \partial \Omega_1 \cap \partial \Omega_2 \), i.e., \( \partial \Omega \cap \partial \Omega_2 \) is a rigid wall boundary. Furthermore, we admit that the electron flow in \( \Omega_2 \) is still thermally conductive, i.e., \( k_e \Omega_2 \geq k_0 > 0 \). Then the natural boundary conditions for \( (P_0) \) at \( \partial \Omega_1 \cap \partial \Omega_2 \) are
\[
\mathbf{v} \cdot \mathbf{n} = 0, \quad \nabla T_e \cdot \mathbf{n} = 0 \quad \text{and} \quad \nabla n_i \cdot \mathbf{n} = 0 \quad (i = 1, \ldots, 6).
\]
The corresponding boundary conditions for \( (P_2) \) at \( \partial \Omega_1 \cap \partial \Omega_2 \) are
\[
\mathbf{v} = 0, \quad T_h = T_{\text{wall}}, \quad \nabla T_e \cdot \mathbf{n} = 0 \quad \text{and} \quad \nabla n_i \cdot \mathbf{n} = 0, \quad i = 1, \ldots, 6.
\]
Due to the geometry of \( \Omega_2 \), the condition
\[
\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{for} \quad (r, z, t) \in (\partial \Omega_1 \cap \partial \Omega_2) \times (0, t_1)
\]
reads
\[
v_r = 0 \quad \text{or} \quad v_z = 0,
\]
depending on the specific part of the boundary.

For \( \partial \Omega_1 \cap \partial \Omega_2 \) we have the same boundary conditions for both problems, \( (P_0) \) and \( (P_2) \). These are presented in detail in [26].

If an outlet is present on \( \partial \Omega_1 \cap \partial \Omega_2 \) with a subsonic outflow, as it is indeed the case for the actual MPD thruster, then for \( (P_0) \) we can prescribe there the pressures: \( p_h = p_e = p_{\text{pump}} \). This is important because \( p_{\text{pump}} \) can be measured in the laboratory experiment.

Remark. Another reduced model, even more appropriate from the numerical point of view, is obtained by neglecting in addition the diffusion terms in the continuity equations (3.1) and in the energy equation (3.26), i.e. the terms involving \( j \).

In this section we analyse an approximate Navier-Stokes/Euler coupled solution obtained by the heterogeneous domain decomposition approach. The analysis is related to the behavior of the inviscid/viscous coupled solution at the interface \( \Gamma \) and provides qualitative information on the quality of the heterogeneous domain decomposition. The computational results show that this method gives reasonably good results although some jumps can be observed at the interface \( \Gamma \) which manifest the need of the implementation of our transition layer correction method for the improvement of the coupled solution also numerically.

As we have seen from our simplified model problem in Chap. 4, the correction method improves the coupled solution significantly and will be employed in our future numerical simulations. For the plasma flow, we are mainly interested in the qualitative behavior of the density, velocity, pressure and temperature associated with the heavy-particle flow along different parts of \( \Gamma \) which are defined by its local properties (supersonic/subsonic, inflow/outflow). Since the numerical results for the pressure and the energy of the heavy-particle flow are relatively accurate across \( \Gamma \), these are eliminated and we construct a subsystem of ODE for the remaining quantities having stronger jumps across \( \Gamma \). We consider a subsystem of (5.55)/(5.59), (5.56) (derived from the conservation of mass); (5.64), (5.65) (derived from the conservation of momentum); and (5.66) (derived from the conservation of energy), describing here the transition layer corrections associated only with the total heavy-particle density \( \rho \), with the components of the velocity field \( \mathbf{v} \), and with the heavy-particle temperature \( T_h \) which depends also on the collision term \( \mathcal{H}_{\text{coll}} \) defined in (3.22), considered here to be a given quantity. It turns out that, at any point \( P \in \Omega_2 \), instead of using the corrections \( c_{0v} \) and \( c_{e0v} \), introduced in (5.35) and (5.37), respectively, it is extremely useful to employ the projections of the correction vector \( \mathbf{c}_v(\tau, \theta, t) \) onto appropriate directions associated with \( P \).

6 Analysis of a Navier-Stokes/Euler coupled solution and transition layer corrections

6.1 Transition layer corrections for the heavy-particle flow

6.1.1 Regularized coupled problem

Since we restrict our considerations to \( \rho, \mathbf{v} \), and \( T_h \), we reformulate the energy equation (3.19). The state equations (3.23) allow us to eliminate the energy \( E_h \) and the pressure \( p_h \) from (3.19), which in terms of \( T_h \) takes in \( \Omega_1 \) the form
\[
\rho c_v \left( \frac{\partial T_h}{\partial t} + v_r \frac{\partial T_h}{\partial r} + v_z \frac{\partial T_h}{\partial z} \right) = \rho c_v \left( \frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial r} - \frac{v_r^2}{r} + \frac{\partial v_z}{\partial z} \right) + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial T_h}{\partial r} + \frac{\partial^2 T_h}{\partial z^2} + \mathcal{H}_{\text{coll}}.
\]

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Here and in what follows, the quantities
\[ \tilde{c}_V := \frac{3}{2} \frac{k}{m_h} \quad \text{and} \quad \tilde{R} := \frac{k}{m_h} \]
define the specific heat at constant volume, and the gas constant for the plasma flow.

The subproblem \( (\tilde{P}_e) \) of the regularized problem \( (P_e) \) is defined by:

1. The continuity equation (3.5) on both subregions \( \Omega_1 \) and \( \Omega_2 \);
2. The momentum equations (3.14), (3.15) in \( \Omega_1 \), and (5.4), (5.5) in \( \Omega_2 \), where the flux functions, the viscous terms and the source terms are defined as in (3.16)–(3.18) and in (5.6)–(5.8), respectively;
3. The energy equation (6.1) in \( \Omega_1 \), and the corresponding equation, obtained by setting \( \mu_h = \kappa_h = \epsilon \) in \( \Omega_2 \):

\[
\rho \tilde{c}_V \left( \frac{\partial h}{\partial t} + v_r \frac{\partial T_h}{\partial r} + v_z \frac{\partial T_h}{\partial z} \right) + \tilde{R} \rho \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = \tau_r \frac{\partial v_r}{\partial r} + \tau_z \frac{\partial v_z}{\partial z} + \epsilon \left[ \frac{\partial^2 T_h}{\partial r^2} + \frac{1}{r} \frac{\partial T_h}{\partial r} + \frac{\partial^2 T_h}{\partial z^2} \right] + \mathcal{H}_{coll}. \tag{6.2}
\]

Also in (6.2), the stress tensor components \( \tau_r, \tau_z \), and \( \tau_{rz} \) are defined as in (5.7).

The source term \( \mathcal{H}_{coll} \) is defined via (3.22) since the continuity equations (3.1) for \( i = 1, \ldots, 6 \) are solvable in the whole domain \( \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma \), and in what follows, will be considered as a given function.

### 6.1.2 Formal asymptotic expansions and new correction functions

In what follows, instead of the corrections \( c_{\rho i}, \ldots, c_{\rho n} \) for all the species densities (as in the regularized problem \( (P_e) \)), we consider here only the correction \( c_\rho \) corresponding to the total density \( \rho \), defined by (5.57). In addition, we consider the correction \( c_{T_h} \) for the heavy-particle temperature \( T_h \). In the far field, the solution vector for the regularized subproblem can be written in the form

\[
\begin{align*}
\rho_{\Omega_2, e}(r, z, t) &= \rho_{\Omega_2}(r, z, t) + c_\rho(e, \theta, t) + \text{rem}_{\rho, \Omega_2, e}(r, z, t), \\
v_{r, \Omega_2, e}(r, z, t) &= v_{r, \Omega_2}(r, z, t) + c_{v_r}(e, \theta, t) + \text{rem}_{v_r, \Omega_2, e}(r, z, t), \\
v_{z, \Omega_2, e}(r, z, t) &= v_{z, \Omega_2}(r, z, t) + c_{v_z}(e, \theta, t) + \text{rem}_{v_z, \Omega_2, e}(r, z, t), \\
T_{h, \Omega_2, e}(r, z, t) &= T_{h, \Omega_2}(r, z, t) + c_{T_h}(e, \theta, t) + \text{rem}_{T_h, \Omega_2, e}(r, z, t). \tag{6.3}
\end{align*}
\]

Note that the coordinates \( (r, z) \in \Omega_2 \) and \( (\epsilon, \theta) \in [0, \infty) \times [0, \theta_{\text{max}}] \) are related to each other by (5.21) and (5.22).

From the transition layer equation (5.56) we get the equation

\[
\left[ \rho + c_\rho(e, \theta, t) \right] \cdot \left[ (v_r \sin \theta + v_z \cos \theta) + (c_{v_r} \sin \theta + c_{v_z} \cos \theta) \right] = \rho [v_r \sin \theta + v_z \cos \theta], \tag{6.4}
\]

where \( \rho, v_r, \) and \( v_z \) are evaluated at the point \( P_0 := (R_0 \sin \theta, R_0 \cos \theta, t) \in \Gamma \times (0, t_1) \) situated on the interface whereas the corrections \( c_{v_r}, c_{v_z}, \) and \( c_{\rho} \) act in the far field at the point

\[
P = P(r, z, t) = P((R_0 + \epsilon \tau) \sin \theta, (R_0 + \epsilon \tau) \cos \theta, t) \quad \text{for} \quad 0 \leq \tau < \infty \tag{6.5}
\]

where \( R_0, \epsilon, \theta, \) and \( t \) are fixed, situated on the radial straight line from the origin \( O \) and passing through \( P_0 \) which corresponds to the fixed angular value \( \theta \in [0, \theta_{\text{max}}] \).

Note that the term \( v_r(P_0) \sin \theta + v_z(P_0) \cos \theta \) in (6.7) represents the normal velocity \( \mathbf{v} \cdot \mathbf{n} \) at \( P_0 \in \Gamma \times (0, t_1) \), see Fig. 9 (left).

Furthermore, at any point \( P(r, z, t) = P(\tau, \theta, t) \) on the straight line, the correction vector

\[
c_\mathbf{v}(\tau, \theta, t) = c_{v_r}(\tau, \theta, t) \mathbf{e}_r + c_{v_z}(\tau, \theta, t) \mathbf{e}_z
\]

associated with the (Euler) velocity field \( \mathbf{v} = (v_r, v_z)(r, z, t) \) can also be represented with respect to the local coordinate system at \( P \) in the form

\[
c_\mathbf{v}(\tau, \theta, t) = c_{v_r}(\tau, \theta, t) \mathbf{e}_r + c_{v_z}(\tau, \theta, t) \mathbf{e}_z
\]

where

\[
\mathbf{e}_r(P) := \mathbf{n}(P_0) = \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_z, \quad \mathbf{e}_r(P) := \mathbf{t}(P_0) = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_z. \tag{6.6}
\]
We now collect the internal layer corrections of order zero corresponding to the above-mentioned flow variables in \( \Omega_c \) and for vector-valued transition layer function and, consequently, the components for all \((5.5), (6.2), \) and collecting the terms of the same (negative) powers of \( \rho \):

Performing the same procedure as in Sect. 5, i.e., inserting the representations (6.3), (6.13), (6.14), and (6.6) into (3.5), (5.4), (6.11)

Note that also the velocity field \( \mathbf{v} \) at the point \( P \) can locally be represented with respect to the coordinate system \((\mathbf{e}_n, \mathbf{e}_t)(P) = (\mathbf{n}, \mathbf{t})(P_0)\) as

and, consequently, the components \((\mathbf{v}\cdot\mathbf{n})_{\Omega_2,w}(r,z,t)\) and \((\mathbf{v}\cdot\mathbf{t})_{\Omega_2,w}(r,z,t)\) of \( \mathbf{v}_{\Omega_2,w}(r,z,t) \) can also be represented in the form

for all \((r,z,t) \in \Omega_2 \times (0,t_1)\). These relations replace now (6.4) and (6.5).

### 6.1.3 Problem for the internal layer corrections

We now collect the internal layer corrections of order zero corresponding to the above-mentioned flow variables in \( \Omega_2 \) in the vector-valued transition layer function

Performing the same procedure as in Sect. 5, i.e., inserting the representations (6.3), (6.13), (6.14), and (6.6) into (3.5), (5.4), (5.5), and (6.2), and collecting the terms of the same (negative) powers of \( \epsilon \) that depend on the rapid variable \( \tau \), we obtain a second-order system of ordinary differential equations for \( \mathbf{c}(\tau, \theta, t) \), which, by integration with respect to the rapid variable \( \tau \), leads to the following system:

\[
\begin{align*}
(c + c_\rho)(\mathbf{v}\cdot\mathbf{n} + c_{\mathbf{v}\cdot\mathbf{n}}) &= \rho \mathbf{v}\cdot\mathbf{n}, \\
\frac{\partial c_{\mathbf{v}\cdot\mathbf{n}}}{\partial \tau} &= \frac{3}{4}(\rho \mathbf{v}\cdot\mathbf{n}) c_{\mathbf{v}\cdot\mathbf{n}} + \frac{3}{4} \tilde{R} [\rho c_{\mathbf{v}\cdot\mathbf{n}} + \frac{3}{4} c_{\mathbf{v}\cdot\mathbf{n}}], \\
\frac{\partial c_{\mathbf{v}\cdot\mathbf{t}}}{\partial \tau} &= (\rho \mathbf{v}\cdot\mathbf{n}) c_{\mathbf{v}\cdot\mathbf{t}},
\end{align*}
\]
Again, the flow quantities $\rho$, $v$, and $T_h$ are evaluated at interface points $P_0$, while the correction vector $\tilde{c}(\tau, \theta, t)$ is considered in the internal layer contained in the far field. In this way, the eqs. (5.64) and (5.65) are replaced by (6.17) and (6.18), while the relation (5.66) is replaced by (6.19).

The complete derivation of this system, together with its analysis will be described in a forthcoming paper [14].

Furthermore, by writing the natural transmission conditions (5.14) in the form

$$\rho_{\Omega_1,\varepsilon} = \rho_{\Omega_2,\varepsilon}, \quad v_{r,\Omega_1,\varepsilon} = v_{r,\Omega_2,\varepsilon}, \quad v_{z,\Omega_1,\varepsilon} = v_{z,\Omega_2,\varepsilon}, \quad T_{h,\Omega_1,\varepsilon} = T_{h,\Omega_2,\varepsilon} \quad \text{on} \quad \Gamma \times (0, t_1),$$

or equivalently,

$$(6.20)_{1,4}, \quad (v \cdot n)_{\Omega_1,\varepsilon} = (v \cdot n)_{\Omega_2,\varepsilon} \quad \text{and} \quad (v \cdot t)_{\Omega_1,\varepsilon} = (v \cdot t)_{\Omega_2,\varepsilon} \quad \text{on} \quad \Gamma \times (0, t_1),$$

and insert (6.3), (6.13), (6.14), and (6.6) into (6.21), we obtain the following initial transmission conditions for the corrections $\tilde{c}(\tau, \theta, t)$ at the value $\tau = 0$ of the rapid variable:

$$c_{\rho}(0, \theta, t) = [\rho_{\Omega_1} - \rho_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right),$$

$$c_{v \cdot n}(0, \theta, t) = [(v \cdot n)_{\Omega_1} - (v \cdot n)_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right),$$

$$c_{v \cdot t}(0, \theta, t) = [(v \cdot t)_{\Omega_1} - (v \cdot t)_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right),$$

$$c_{T_h}(0, \theta, t) = [T_{h,\Omega_1} - T_{h,\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right).$$

Note that the right-hand sides of (6.22) are nothing else but the jumps of the coupled Navier-Stokes/Euler solution components at the interface $\Gamma$.

Finally, we complete the problem for the corrections with the condition at infinity:

$$\lim_{\tau \to \infty} \tilde{c}(\tau, \theta, t) = 0.$$  \hspace{1cm}(6.23)

### 6.1.4 Solution of the subproblem for $\tilde{c}(\tau, \theta, t)$

Compared with the ODE system (5.55)/(5.59), (5.56), (5.64), (5.65), and (5.66), the system (6.16)–(6.19) has a compact form. It has to be solved with respect to the rapid variable $\tau \geq 0$ for any time $t \in (0, t_1)$ and at any interface point $(R_0 \sin \theta, R_0 \cos \theta)$ defined by the direction $\theta \in [-\theta_{\max}, \theta_{\max}]$. A complete analysis of this system, together with the method of improving the Navier-Stokes/Euler solution by using the correction subvector $\tilde{c}(\tau, \theta, t)$ from (6.15) will be presented in [14]. There, the correction functions $c_{\rho}$, $c_{v \cdot n}$, $c_{v \cdot t}$, and $c_{T_h}$ again are considered as flags, indicating whether and perhaps which continuity conditions have to be employed at $(R_0 \sin \theta, R_0 \cos \theta, t) \in \Gamma \times (0, t_1)$.

We emphasize that the solution of the initial value problem (6.18), (6.22) for $c_{v \cdot t}(\tau, \theta, t)$ can be calculated analytically as

$$c_{v \cdot t}(\tau, \theta, t) = [(v \cdot t)_{\Omega_1} - (v \cdot t)_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right) \cdot e^{(\rho v \cdot n)(R_0 \sin \theta, R_0 \cos \theta, t) \cdot \tau}$$

for $\tau \geq 0$. As the transition layer term $c_{v \cdot t}$ should vanish for $\tau \to \infty$,

$$\lim_{\tau \to \infty} c_{v \cdot t}(\tau, \theta, t) = 0,$$  \hspace{1cm}(6.25)

we can distinguish two cases:

1. $(\rho v \cdot n)(R_0 \sin \theta, R_0 \cos \theta, t) < 0$:

   In this case, corresponding to a plasma flow leaving the Euler subregion and entering the Navier-Stokes one, the condition (6.25) is satisfied independent of the coefficient $[(v \cdot t)_{\Omega_1} - (v \cdot t)_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right)$. Therefore, the correction $c_{v \cdot t}(\tau, \theta, t)$ exists for all $\tau \geq 0$, and the velocity component $v(P) \cdot e_\theta$ at any point in $\Omega_2$ has to be “corrected” by $c_{v \cdot t} \rho$ via (6.14).

2. $(\rho v \cdot n)(R_0 \sin \theta, R_0 \cos \theta, t) \geq 0$:

   In this case, corresponding to a plasma flow entering the Euler subregion, or flowing in tangential direction along $\Gamma$ at $P_0$, the condition at infinity (6.25) is satisfied only in the case

   $$[(v \cdot t)_{\Omega_1} - (v \cdot t)_{\Omega_2}] \left( R_0 \sin \theta, R_0 \cos \theta, t \right) = 0,$$

   which implies $c_{v \cdot t}(\tau, \theta, t) = 0$ for all $\tau \geq 0$. Therefore, in addition to the flux conditions, we have to impose also the continuity of the tangential velocity as one of the transmission conditions at the interface point $P_0$. 

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6.2 Analysis of a Navier-Stokes/Euler coupled solution

In this subsection we present the behavior of a Navier-Stokes/Euler coupled solution obtained by the heterogeneous domain decomposition approach. In the numerical code, the discretization of the extended conservation laws (3.5), (3.14), (3.15), and (3.19), describing the heavy-particle flow, of the energy equation (3.30) for the electron field, and of the discharge equation (3.34) is carried out by the use of a second-order finite volume upwind scheme based on a parameter-controlled flux vector splitting combined with an explicit Euler time-stepping. The scheme is extended for the heterogeneous coupling, whereby a weighted essentially non-oscillatory (WENO) reconstruction is also employed for the calculation of the gradients across the coupling boundary $\Gamma$ (for more details, see [9]). The continuity of the normal fluxes is imposed as a transmission condition across the interface.

In Fig. 10 we plot the (primal) triangular mesh and the corresponding dual mesh used for the numerical finite volume treatment of the coupled problem. The mesh generator, based on an advancing front algorithm, enforces the global mesh to be conforming at the artificial coupling boundary.

Fig. 11 presents the distribution of the velocity field $v$ in some vicinity of the interface $\Gamma$. The arrows indicate the orientation, while the colours indicate the magnitude of the velocity vector. A high-velocity, supersonic central plasma jet leaves $\Omega_1$ and enters $\Omega_2$ with Mach numbers of about 4, as will be seen later. In addition, we observe a recirculation of the plasma flow in the vicinity of the left (insulator) boundary situated on top of the anode. This recirculation motivated our heterogeneous domain decomposition approach, because in that region the boundary $\Gamma$ obtains “inflow” properties for $\Omega_1$ and requires additional information on the flow quantities.

Fig. 12 shows the isolines of the longitudinal velocity $v_z$ and of the radial velocity $v_r$ for the coupled solution. Obviously, the coupling method works very well for the central, hot plasma jet. Up to one nozzle radius above the centerline, $v_z$ passes smoothly the coupling boundary $\Gamma$. However, farther away from the centerline, the isolines show small jumps and have kinks across the interface. The radial velocity component $v_r$ also passes smoothly the coupling boundary $\Gamma$ in the central plasma jet,
but this property is lost as we approach the upper part of the coupling boundary. Furthermore, a shock wave can be observed in $\Omega_2$, in the immediate vicinity of $\Gamma$.

Fig. 13 shows the isolines of the heavy-particle temperature $T_h$ on the left, and the values of this quantity taken from the Navier-Stokes and the Euler regions, respectively, along the coupling boundary (right). While the isolines of $T_h$ behave smoothly across the part of $\Gamma$ contained in the central plasma jet, we can very clearly see the discontinuities of the solution in the region above. The behavior of $T_h$ at the left (insulator) boundary on top of the anode is due to the different boundary conditions imposed on $T_h$: Dirichlet conditions on the near-field part and homogeneous Neumann conditions in the Euler far-field part at the insulator boundary. A shock wave can clearly be observed in the immediate vicinity of $\Gamma$ in $\Omega_2$. The discontinuities of $T_h$ across $\Gamma$ show that the heavy-particle heat conduction is still physically relevant with respect to the inviscid Euler energy flux, such that the coupled solution has to be corrected in such a way, that the continuity of $T_h$ across $\Gamma$ is reobtained, and that the normal heat flux of the heavy-particle flow is conserved across the coupling boundary.
Fig. 15  Isolines of the electron temperature (left); electron temperature distribution along the coupling boundary (right).

Fig. 16  Isolines of the Mach number (left); Mach number distribution along the coupling boundary (right).

Fig. 14 contains the isolines (left) of the heavy-particle pressure $p_h$ and the behavior of $p_h$ across $\Gamma$. Also behaves well across $\Gamma$ in the central plasma jet, and becomes discontinuous in the region above, but the jumps are not so severe as for $T_h$. This is a reason for considering the correction system in $\rho, \mathbf{v} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{t}$, and $T_h$.

On the other hand, the approximate coupled solution also shows that the electron temperature $T_e$ passes the artificial interface smoothly, as can be seen in Fig. 15. Thus, the natural transmission condition for the electron field is justified also numerically. The very small jumps which can be observed in the recirculation zone are a consequence of the collision phenomena which take place between different species of the plasma flow there.

In Fig. 16, we plot the isolines of the Mach number associated with the velocity $\mathbf{v}$ (left), and the Mach number distribution corresponding to the normal velocity $\mathbf{v} \cdot \mathbf{n}$ at $\Gamma$:

$$M_{\mathbf{v} \cdot \mathbf{n}} := \frac{\mathbf{v} \cdot \mathbf{n}}{\sqrt{\gamma \frac{p_h + p_e}{\rho}}} \quad (6.26)$$

Fig. 17 shows the profiles of the normal and the tangential velocity components $\mathbf{v} \cdot \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{t}$ calculated from the Navier-Stokes and Euler subregions, respectively. While the jumps in $\mathbf{v} \cdot \mathbf{n}$ are uniformly small (due to the conservativity of the numerical scheme applied), the tangential velocity becomes rather discontinuous, especially in the recirculation region.

Finally, Fig. 18 contains the normal and tangential mass fluxes $(\rho \mathbf{v} \cdot \mathbf{n})_\Omega$ and $(\rho \mathbf{v} \cdot \mathbf{t})_\Omega$ ($i = 1, 2$) across the coupling interface.

7 Concluding comments

Based on the experimentally justified fact that the viscosity effects and the magnetic field are negligible far from the thruster, we can split the flow field into two parts by an artificial internal boundary $\Gamma$, thus reducing the computational cost by associating a simplified model with the far field. In our treatment $\Gamma$ is fixed. Then the question arises, how far this $\Gamma$ should be chosen. The more far the better, but the computational cost then is increasing when the full model occupies a larger domain. So, we have to find an “optimal” position of $\Gamma$. In addition, we must take into account the inflow properties of the plasma which means that the model of the flow changes at some parts of $\Gamma$. We introduce here a new method for associating transmission conditions. This method is based on singular perturbation theory. In addition, we can approximate by this method the solution of the
perturbed model $M_\epsilon$ (which is “close” to the real physical model) by the solution of the reduced model $M_0$, plus corresponding corrections (if any). Our numerical simulations illustrate the theoretical results.

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References


