Stability for a damped nonlinear oscillator

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Abstract

The stability of the null solution of Eq. (1.1) below is discussed. Under some unusual assumptions, we obtain new stability results for this classical oscillator equation. Our approach allows extensions to both the vector case and the case of the whole real line.

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1. Introduction

Consider the second-order ODE

\[ x'' + 2f(t)x' + \beta(t)x + g(t, x) = 0, \quad t \in \mathbb{R}_+, \] \hfill (1.1)

where $\mathbb{R}_+ := [0, +\infty)$, $f, \beta : \mathbb{R}_+ \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are three given functions.

The most familiar interpretation of this equation is that it describes nonlinear damped oscillations. Stability problems for this ODE have been studied intensively so far (see, e.g., [3–6,10–12], and the references therein). Recently, Burton and Furumochi [1] have introduced a new method to study the stability of the null solution $x = x' = 0$ of Eq. (1.1),

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which is based on the Schauder fixed-point theorem. They discuss a particular case of
(1.1) (one of their assumptions is $\beta(t) = 1$) to illustrate their technique. In [8] we prove
stability results for the null solution of the same equation by using relatively classical
arguments. Here, we reconsider Eq. (1.1) under more general assumptions, which require
more sophisticated arguments, and prove stability results (see Theorem 2.1 below). In
particular, we obtain the generalized exponential asymptotic stability of the trivial solution.
See [7, p. 158] for the definition of this concept.

Our approach allows extensions to both the vector case and the case $t \in \mathbb{R}$. Notice that
g(t, \cdot) does not necessarily have to be of the gradient type, which is the case investigated
recently by Pucci and Serrin [10–12].

2. The main result

The following hypotheses will be required:

(i) $f \in C^1(\mathbb{R}_+)$ and $f(t) \geq 0$ for all $t \geq 0$;
(ii) $\int_0^{+\infty} f(t) \, dt = +\infty$;
(iii) there exist two constants $h$, $K \geq 0$ such that

$$|f'(t) + f^2(t)| \leq K f(t), \quad \forall t \in [h, +\infty),$$

(iv) $\beta \in C^1(\mathbb{R}_+)$, $\beta$ is decreasing, and

$$\beta(t) \geq \beta_0 > K^2, \quad \forall t \in \mathbb{R}_+,$$

where $\beta_0$ is a constant;
(v) $g \in C(\mathbb{R}_+ \times \mathbb{R})$ and $g$ is locally Lipschitzian in $x$;
(vi) $g$ satisfies the following estimate

$$|g(t, x)| \leq f(t) \sigma(|x|), \quad \forall t \in \mathbb{R}_+,$$

where $\sigma$ denotes the usual Landau symbol.

These assumptions are inspired by those in [1], but are more general. Notice that (i) and
(iii) imply that $f$ is uniformly bounded (see [8, Remark 2.2]).

The main result of this paper is the following theorem:

Theorem 2.1. If assumptions (i), (iii)–(vi) are fulfilled, then the null solution to (1.1) is
uniformly stable. If in addition (ii) holds, then the null solution to (1.1) is asymptotically
stable.

Remark 2.1. Under assumptions (i)–(vi), we cannot expect to have uniform asymptotic
stability for the null solution.

Indeed, even in the case $g = 0$ and $\beta = \text{Const.}$, say $\beta(t) = 1$, $\forall t \in \mathbb{R}_+$, one can construct
a fundamental matrix $X(t)$ for the corresponding first-order linear differential system in
\( (x, \ y = x') \), for which \( \|X(t)X(t)\|^{-1} \) does not converge to zero as \( t - \tau \to \infty \) (here \( \| \cdot \| \) denotes a matrix norm) (see [8, Remark 2.3]).

For fundamental concepts and results in stability theory we refer the reader to [2,7,9].

3. Proof of Theorem 2.1

As in [1], we write Eq. (1.1) as the following first-order system:

\[
\begin{align*}
\dot{z}' &= A(t)z + B(t)z + F(t, z),
\end{align*}
\]

(3.1)

where

\[
\begin{align*}
z &= \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} -f(t) & 1 \\ -\beta(t) & -f(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix}, \\
F(t, z) &= \begin{pmatrix} 0 \\ -g(t, x) \end{pmatrix}.
\end{align*}
\]

It is easily seen that our stability question reduces to the stability of the null solution \( z(t) = 0 \) to system (3.1).

Let \( t_0 \geq 0 \) be arbitrarily fixed and let

\[
Z(t, t_0) := \begin{pmatrix} a(t, t_0) & b(t, t_0) \\ c(t, t_0) & d(t, t_0) \end{pmatrix},
\]

\( t \geq t_0 \), be the fundamental matrix to the linear system

\[
\begin{align*}
\dot{z}' &= A(t)z,
\end{align*}
\]

(3.2)

which is equal to the identity matrix for \( t = t_0 \).

Then

\[
\begin{align*}
a'(t, t_0) &= -f(t)a(t, t_0) + c(t, t_0), \\
b'(t, t_0) &= -\beta(t)a(t, t_0) - f(t)c(t, t_0), \\
c'(t, t_0) &= -\beta(t)b(t, t_0) - f(t)d(t, t_0), \\
d'(t, t_0) &= -f(t)b(t, t_0) - f(t)d(t, t_0).
\end{align*}
\]

(3.3)

So, since \( \beta \) is decreasing (hypothesis (iv)), the first two equations of (3.3) leads us to

\[
\frac{1}{2} (\beta(t)a(t, t_0)^2 + c(t, t_0)^2)' \leq -f(t)(\beta(t)a(t, t_0)^2 + c(t, t_0)^2)
\]

and hence

\[
\beta(t)a(t, t_0)^2 + c(t, t_0)^2 \leq \beta(t_0) e^{-\int_{t_0}^{t} f(u) \, du}, \quad \forall t \geq t_0.
\]

(3.4)

Similarly, from the last two equations of (3.3), we get

\[
\beta(t)b(t, t_0)^2 + d(t, t_0)^2 \leq e^{-\int_{t_0}^{t} f(u) \, du}, \quad \forall t \geq t_0.
\]

(3.5)

Consider for \( z = (x, y)^T \in \mathbb{R}^2 \) the norm \( \|z\| := \sqrt{x^2 + y^2} \).
For \( z_0 = (x_0, y_0)^\top \in \mathbb{R}^2 \) we obtain from (3.4), (3.5), and hypothesis (iv),

\[
\|Z(t, t_0)z_0\| = \left\| \begin{pmatrix}
  a(t, t_0)x_0 + b(t, t_0)y_0 \\
  c(t, t_0)x_0 + d(t, t_0)y_0
\end{pmatrix}
\right\| \leq \sqrt{x_0^2 + y_0^2}
\times \sqrt{\beta(t)[a(t, t_0)^2 + b(t, t_0)^2] + [c(t, t_0)^2 + d(t, t_0)^2]}
\leq \gamma \sqrt{1 + \beta(t_0)}e^{-\int_{t_0}^{t} f(u) \, du} \|z_0\|,
\]

where \( \gamma = \max\{1, 1 / \sqrt{\beta_0}\} \).

Moreover, since

\[
Z(t, t_0)Z(s, t_0)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta(t, s, t_0) \\ \mu(t, s, t_0) \end{pmatrix}, \quad \forall t \geq s \geq t_0 \geq 0,
\]

satisfies system (3.2), we deduce as before

\[
\left\| Z(t, t_0)Z(s, t_0)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \sqrt{\beta_0 \eta(t, s, t_0)^2 + \mu(t, s, t_0)^2}
\leq \sqrt{\beta(t) \eta(t, s, t_0)^2 + \mu(t, s, t_0)^2}
\leq \sqrt{\beta(s) \eta(s, s, t_0)^2 + \mu(s, s, t_0)^2}e^{-\int_{s}^{t} f(u) \, du}
= e^{-\int_{t_0}^{t} f(u) \, du}, \quad \forall t \geq s \geq t_0 \geq 0.
\]

Let us prove the first part of Theorem 2.1. Consider \( z_0 \neq 0 \) with \( \|z_0\| \) small enough, \( t_0 \geq 0 \), and let us denote by \( z(t, t_0, z_0) \) the unique solution to (3.1) which is equal to \( z_0 \) for \( t = t_0 \). By hypotheses (i) and (v), \( z(t, t_0, z_0) \) is defined on a maximal right interval, say \( [t_0, l] \), and satisfies the following integral equation:

\[
z(t, t_0, z_0) = Z(t, t_0)z_0 + \int_{t_0}^{t} Z(t, t_0)Z(s, t_0)^{-1} [B(s)z(s, t_0, z_0)
\quad + F(s, z(s, t_0, z_0))] \, ds,
\]

for all \( t \in [t_0, l] \).

By (3.6)–(3.8) we infer that

\[
\|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(t_0)}\|z_0\|e^{-\int_{t_0}^{t} f(s) \, ds}
\quad + \int_{t_0}^{t} e^{-\int_{s}^{t} f(u) \, du} \|f'(s) + f^2(s)\| |x(s, t_0, z_0)|
\quad + |g(s, x(s, t_0, z_0))| \, ds,
\]

for all \( t \in [t_0, l] \). In fact, for our proof, we can replace \( g \) by another function, say \( \tilde{g} \), defined as follows. Take some

\[
\theta \in (0, \sqrt{\beta_0} - K).
\]

By hypothesis (vi) it follows that there exists a \( \rho > 0 \) such that if \( |x| < \rho \), then

\[
|g(t, x)| \leq \theta f(t)|x|.
\]
Now, we define the function $\tilde{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by
\[
\tilde{g}(t, x) := \begin{cases} 
g(t, x) & \text{if } |x| < \rho, 
g(t, \rho) & \text{if } x \geq \rho, 
g(t, -\rho) & \text{if } x \leq -\rho 
\end{cases}
\]
for all $t \geq 0$.

It is readily seen that for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,
\[
|\tilde{g}(t, x)| \leq \theta f(t)|x|,
\]
$\tilde{g}$ is of class $C(\mathbb{R}_+ \times \mathbb{R})$, and is locally Lipschitzian in $x$. With this in mind, we shall admit from now on that the original function $g$ satisfies all the properties of $\tilde{g}$. We are going to show that $l = \infty$. First, let us assume that $0 \leq t_0 < h$. We admit, by contradiction, that $l$ is finite. If $l \leq h$, since $f \in C^1[0, h]$, we infer from (3.9) that
\[
\|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(t_0)}\|z_0\| + D \int_{t_0}^{t} \|z(s, t_0, z_0)\| \, ds, \quad \forall t \in [t_0, l),
\]
with a positive constant $D$. Therefore (see, e.g., [7, p. 37]),
\[
\|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(t_0)}e^{Dh}\|z_0\|, \quad \forall t \in [t_0, l) .
\]
(3.11)

Thus, $z(t, t_0, z_0)$ as well as $z'(t, t_0, z_0)$ are bounded on $[t_0, l)$ and so $z(t, t_0, z_0)$ can be extended to the right of $l$. This fact contradicts the maximality of $l$. Therefore $z(t, t_0, z_0)$ exists on $[t_0, l)$, with $l > h$.

Now, we assume $h < l < \infty$. We are going to find an estimate for $z(t, t_0, z_0)$ on the interval $[h, l)$. This time, our hypothesis (iii) comes into play. We have for all $t \in [h, l)$
\[
\|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(h)}\|z(h, t_0, z_0)\|e^{-\int_{h}^{t} f(u) \, du} \\
+ \int_{h}^{t} e^{-\int_{h}^{s} f(u) \, du} K f(s) |x(s, t_0, z_0)| \, ds \\
+ \int_{h}^{t} e^{-\int_{h}^{s} f(u) \, du} |g(s, x(s, t_0, z_0))| \, ds.
\]
(3.12)

From (3.12) it follows that
\[
\|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(h)}\|z(h, t_0, z_0)\|e^{-\int_{h}^{t} f(u) \, du} \\
+ \frac{K + \theta}{\sqrt{\beta_0}} \int_{h}^{t} e^{-\int_{s}^{h} f(u) \, du} f(s) \|z(s, t_0, z_0)\| \, ds \\
=: v(t), \quad \forall t \in [h, l).
\]
(3.13)

Then, by (3.13), straightforward calculations lead us to
\[
v'(t) \leq \left(\frac{K + \theta}{\sqrt{\beta_0}} - 1\right) f(t)v(t), \quad \forall t \in [h, l),
\]
\[
v(h) = \gamma \sqrt{1 + \beta(h)}\|z(h, t_0, z_0)\|,
\]
and so
\[ \|z(t, t_0, z_0)\| \leq v(t) \leq v(h)e^{((K+\theta)/\sqrt{\beta_0-1}) \int_{t_0}^t f(u) \, du}, \quad \forall t \in [h, l). \] (3.14)
From (3.14) we can see that \( z(t, t_0, z_0) \) is bounded. Since \( z'(t, t_0, z_0) \) is also bounded, it follows that \( l = \infty \). Now, for \( \varepsilon > 0 \) we denote
\[ \delta = \delta(\varepsilon) = \frac{\varepsilon e^{-Dh}}{\gamma^2 \sqrt{1 + \beta(0)}(1 + \beta(h))}. \]
From (3.11) it follows that \( \|z(t, t_0, z_0)\| < \varepsilon/\gamma \sqrt{1 + \beta(h)} \) for all \( t \in [t_0, h] \), provided that \( \|z_0\| < \delta \). Therefore, by (3.10) and (3.14), \( \|z(t, t_0, z_0)\| < v(h) < \varepsilon \) for all \( t \geq h \). Summarizing, if \( 0 \leq t_0 < h \), the solution \( z(t, t_0, z_0) \) starting from any point \( z_0 \), with \( \|z_0\| < \delta \), exists on \([t_0, \infty)\) and satisfies \( \|z(t, t_0, z_0)\| < \varepsilon \) for all \( t \geq t_0 \).

If \( t_0 \geq h \), then analogously we obtain that \( l = \infty \) and
\[ \|z(t, t_0, z_0)\| \leq \gamma \sqrt{1 + \beta(t_0)} \|z_0\| e^{((K+\theta)/\sqrt{\beta_0-1}) \int_{t_0}^t f(u) \, du}, \quad \forall t \in [h, \infty). \] (3.15)
Therefore, with the same \( \delta \) as before, \( \|z_0\| < \delta \) implies again \( \|z(t, t_0, z_0)\| < \varepsilon \) for all \( t \geq t_0 \).

Hence the null solution is uniformly stable. If in addition (ii) is fulfilled, then by (3.10), (3.14), and (3.15) it follows that the null solution to (1.1) is asymptotically stable. Now it is clear that replacing \( g \) by \( \tilde{g} \) is allowed by the fact that for small \( \|z_0\| \) the values of \( x(t, t_0, z_0) \) remain in the interval \((-\rho, \rho)\). The proof of Theorem 2.1 is complete. \( \square \)

**Remark 3.1.** If \( f \) satisfies (i)–(iii), then \( f(t) > 0 \) for all \( t \geq h \). Let us assume, by contradiction, that \( f(t_1) = 0 \) for some \( t_1 \geq h \). Then, one can prove that \( f(t) = 0 \) for all \( t \geq t_1 \). Indeed, if \( f(t_2) > 0 \) for some \( t_2 > t_1 \), then the function \( u = 1/f \) is well defined on the maximal interval containing \( t_2 \) on which \( f > 0 \), say \((c, d)\), and satisfies the inequality
\[ u'(t) + Ku(t) - 1 \geq 0, \quad t \in (c, d). \]
This implies that
\[ \frac{d}{dt}(e^{Kt}[u(t) - 1/K]) \geq 0, \quad t \in (c, t_2], \]
i.e., the function \( t \mapsto e^{Kt}[u(t) - 1/K] \) is nondecreasing on \((c, t_2]\). But this is impossible since the limit of this function as \( t \to +\infty \) is \( +\infty \). Thus, \( f(t) = 0 \) for all \( t \geq t_1 \), which contradicts (ii). Therefore, we have proved that indeed \( f(t) > 0 \) for all \( t \geq h \). Consequently, the function \( p(t) = \int_0^t f(s) \, ds \) is strictly increasing, at least for \( t \geq h \). By the above proof we have that the null solution is generalized exponentially asymptotically stable (see [7, p. 158]), with this \( p(t) \) if \( f(t) > 0 \) on \( \mathbb{R}_+ \), and with a modified \( p(t) \) which is a strictly increasing function on \( \mathbb{R}_+ \), for instance,
\[ p(t) = \begin{cases} f(h)t & \text{if } 0 \leq t \leq h, \\ f(h)h + \int_h^t f(s) \, ds & \text{if } t > h, \end{cases} \]
if \( h > 0 \) and \( f \) vanishes on some subsets of the interval \([0, h)\).
Remark 3.2. While the classical transformation \((x := x, y := x')\) is useless when trying to obtain stability results for the null solution to Eq. (1.1), the transformation (3.1), introduced in [1], is essential in deriving our estimates on the solution.

Remark 3.3. In [1,8] the following assumption is required instead of (vi)

\[ |g(t, x)| \leq Mf(t)|x|^\alpha, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \]

where \(M > 0\) and \(\alpha > 1\) are constants. Obviously, this is a particular case of (vi), where \(o(r) = Mr^\alpha\). Assuming in addition that \(\beta(t) = 1\) we found in [8] the following estimate:

\[
\|z(t, t_0, z_0)\| \leq \left\{ \begin{array}{l}
e^{(\alpha-1)(1-K) \int_{t_0}^t f(s) \, ds} \left[ \|z_0\|^{1-\alpha} - \frac{M}{1-K} \right] \\
+ \frac{M}{1-K} \end{array} \right\}^{1/(1-\alpha)}, \tag{3.16}
\]

for all \(t \geq t_0 \geq h\) and \(\|z_0\|\) small, which is stronger than (3.15). Indeed, (3.16) implies

\[
\|z(t, t_0, z_0)\| \leq e^{(K-1) \int_{t_0}^t f(s) \, ds} \left[ \|z_0\|^{1-\alpha} - \frac{M}{1-K} \right]^{1/(1-\alpha)},
\]

with the better exponent \(K - 1\).

Remark 3.4. If \(\beta(t) = 1, \forall t \in \mathbb{R}_+\), the fundamental matrix \(Z(t, t_0)\) can be determined explicitly (see [1,8]),

\[
Z(t, t_0) = e^{-\int_{t_0}^t f(u) \, du} \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}.
\]

In general, this is not possible, so in our proof we had to get estimates without having an explicit form of \(Z(t, t_0)\).

Remark 3.5. It is obvious that our stability results can be extended to the case when (1.1) is a vectorial equation. More precisely, let us assume that \(f\) and \(\beta\) are scalar functions satisfying the same conditions as before, while \(x = (x_1, \ldots, x_N)^T : \mathbb{R}_+ \to \mathbb{R}^N\), and \(g = (g_1, \ldots, g_N)^T : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N\) such that:

(v) \(g \in C(\mathbb{R}_+ \times \mathbb{R}^N)\) and \(g\) is locally Lipschitzian in \((0, \ldots, 0)\);

(vii) \(g\) satisfies the following estimate

\[
\|g(t, x)\| \leq Mf(\|x\|), \quad \forall t \in \mathbb{R}_+,
\]

where \(\| \cdot \|\) denotes some norm in \(\mathbb{R}^N\). Under the above assumptions Theorem 2.1 holds for the corresponding vectorial form of Eq. (1.1).

The proof is similar, with the function \(\tilde{g} : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N\) defined by

\[
\tilde{g}(t, x) := g(t, P_{B_\rho} x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,
\]

where \(P_{B_\rho}\) denotes the projection operator on the ball

\[
B_\rho := \{x \in \mathbb{R}^N, \|x\| \leq \rho\}.
\]
Unlike [10], here g does not necessarily have to be the gradient of a scalar potential.

**Remark 3.6.** Clearly, our stability theorem can be reformulated for a nonlinear function \( g \) which is defined only for \( x \) belonging to a ball centered at the origin. Moreover, once \( \beta_0 \) and \( K \) are given, it is enough to assume that, for all \( t \geq 0 \) and \( x \) in the given ball, we have \( \|g(t, x)\| \leq \theta f(t)\|x\| \), with some \( \theta > 0 \) small enough.

**Remark 3.7.** Our results can also be extended to the whole \( \mathbb{R} \), as in [8].

**Example 3.1.** Here are some examples of typical functions \( f, \beta, g \) which satisfy the assumptions (i)–(vi):

\[
f(t) = \frac{1}{t} \quad \text{or} \quad f(t) = \frac{1}{t \ln t}, \quad \forall t \geq h > 0,
\]

\[
\beta(t) = 1 + e^{-t}, \quad g(t, x) = f(t)x^\alpha, \quad \alpha > 1,
\]

where \( f \) is extended to a smooth nonnegative function defined on \( \mathbb{R}_+ \).

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