Stability for a Nonlinear Second Order ODE

By

Gheorghe Moroşanu and Cristian Vladimirescu

(Central European University, Hungary and University of Craiova, Romania)

Abstract. The stability of the null solution of the equation (1.1) below is investigated. The main arguments are based on some Bernoulli type differential inequalities. Extensions to the whole real line are also discussed.

Key Words and Phrases. Stability, Bernoulli type differential inequalities.

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1. Introduction

Consider the equation

\[ x'' + 2f(t)x' + x + g(t, x) = 0, \quad t \in R_+, \]

where \( R_+ := [0, +\infty) \), \( f : R_+ \to R_+ \) and \( g : R_+ \times R \to R \) are two given continuous functions.

It is well known that (1.1) is a mathematical model for (nonlinear) damped oscillatory phenomena. Much work has been dedicated so far to the stability questions for such kind of equations (see, e.g., [3–6], [8–10] and the references therein). In particular, the asymptotic stability of the null solution of equation (1.1) has been studied in the recent paper [1] by means of a new approach based on the Schauder Fixed Point Theorem.

In the present Note some stability results are proved under assumptions more general than those of [1] (see Remark 2.1 in Section 2). Our approach is based on elementary arguments only, involving in particular some Bernoulli type differential inequalities.

Under our assumptions \( f \) can be chosen in a larger class of functions, which in particular allows extensions of our stability results to the whole real line \( R \) (see Section 4).

2. The main result

The following hypotheses will be required:

(i) \( f \in C^1(R_+) \) and \( f'(t) > 0 \), for all \( t > 0 \);

(ii) \( \int_0^{+\infty} f(t)dt = +\infty \);
(iii) there exist $a \geq 0$ and $K \in (0, 1)$ such that
\begin{equation}
|f'(t) + f^2(t)| \leq Kf(t), \quad \text{for all } t \in [a, +\infty);
\end{equation}
(iv) $g \in C(R_+ \times R)$ and $g$ is locally Lipschitzian in $x$;
(v) there exist $M > 0$ and $\alpha > 1$ such that
\begin{equation}
|g(t, x)| \leq Mf(t)|x|^{\alpha}, \quad \text{for all } (t, x) \in R_+ \times R.
\end{equation}

An example of functions $f$ and $g$ is (cf. [1])
\[ f(t) = \frac{1}{t+1}, \quad g(t, x) = f(t) \cdot x^{\alpha}. \]

Indeed, these functions fulfil (i)–(v) with $a \geq 0$, $K \in (0, 1)$, $\alpha > 1$ arbitrary, and $M = 1$. See also the example given in Section 4.

**Remark 2.1.** In [1] the following additional assumptions are required: $f(t) \to 0$ as $t \to \infty$; the constant $a$ in (iii) is fixed to $a = 0$; a more restrictive condition is assumed instead of (iv), namely
\[ |g(t, x) - g(t, y)| \leq L(\delta)f(t)|x - y| \quad \forall t \geq 0, \ |x|, \ |y| \leq \delta, \]
with $L(\delta)$ continuous and increasing.

**Remark 2.2.** If either $a > 0$ or $a = 0$ and $f(0) > 0$, we can easily derive from (2.1) that the function $u = 1/f$ satisfies
\[ -Ku(t) \leq 1 - u'(t) \leq Ku(t) \quad \forall t \geq a. \]

After an easy computation we see that $f$ necessarily satisfies
\[ \frac{K}{1 + e^{K(t-a)}(1 + K/f(a))} \leq f(t) \leq \frac{K}{1 + e^{-K(t-a)}(1 + K/f(a))} \quad \forall t \geq a. \]

If $a = 0$ and $f(0) = 0$ we can derive the same estimates on any interval $[a_1, \infty)$, $a_1 > 0$. In particular, the second estimate implies that $f$ is uniformly bounded: there exists a $c > 0$ such that $0 < f(t) \leq c \ \forall t \geq 0$ (i.e., we are in the case of “small damping” (see, e.g., [6], p. 415)).

Recall that the null solution of equation (1.1) is said to be stable (in the sense of Lyapunov) if for every $\varepsilon > 0$ and $t_0 \geq 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x(t_0)| < \delta$ and $|x'(t_0)| < \delta$ imply that $x(t)$ exists on $[t_0, +\infty)$ and $|x(t)| < \varepsilon$, $|x'(t)| < \varepsilon$ for all $t \geq t_0$. If $\delta$ depends on $\varepsilon$ only (i.e., it is independent of $t_0$), then the null solution is said to be uniformly stable. The null solution is
called *asymptotically stable* if it is stable and in addition we have: for every \( t_0 \geq 0 \) there exists a \( \mu = \mu(t_0) > 0 \) such that both \( x(t) \) and \( x'(t) \) tend to zero as \( t \to +\infty \) whenever \( |x(t_0)| < \mu(t_0) \) and \( |x'(t_0)| < \mu(t_0) \). For the definitions of different concepts of stability and for standard stability theory see, e.g., [2] and [7]. The main result of this Note is the following theorem.

**Theorem 2.1.** If assumptions (i), (iii), (iv) and (v) are fulfilled, then the null solution of equation (1.1) is uniformly stable. If in addition (ii) holds, then the null solution of (1.1) is asymptotically stable.

**Remark 2.3.** Under our assumptions (i)–(v), we cannot expect to have uniform asymptotic stability for the null solution. Indeed, this can be illustrated by the following simple counter example: \( f(t) = 1/(t + 1) \) and \( g = 0 \). The general solution of (1.1) is given by

\[
x(t) = (t + 1)^{-1}(C_1 \cos(t + 1) + C_2 \sin(t + 1)),
\]

which allows us the construction of a fundamental matrix \( X(t) \) for the corresponding first order linear differential system in \((x, y = x')\). The null solution of this system is not uniformly asymptotically stable since \( ||X(t)X^{-1}(\tau)||, \quad t \geq \tau \geq 0 \) does not converge to 0 as \( t - \tau \to \infty \) (Recall that (see, e.g., [2], p. 92) a necessary and sufficient condition for the null solution to be uniformly asymptotically stable is that \( ||X(t)X^{-1}(\tau)|| \leq Me^{-\beta(t-\tau)}, \quad t \geq \tau \geq 0 \) for some positive constants \( M \) and \( \beta \).

3. **Proof of Theorem 2.1**

As in [1], we write equation (1.1) as the following first order system

\[
z' = A(t)z + B(t)z + F(t, z),
\]

where

\[
z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} -f(t) & 1 \\ 1 & -f(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ f'(t) + f^2(t) \end{pmatrix}, \quad F(t, z) = \begin{pmatrix} 0 \\ -g(t, x) \end{pmatrix}.
\]

Obviously, our stability question reduces to the stability of the null solution \( z(t) = 0 \) of system (3.1). Notice that the fundamental matrix of the linear system

\[
z'(t) = A(t)z,
\]

which is equal to the identity matrix for \( t = t_0, \quad t_0 \geq 0 \), is given by
\[ Z(t, t_0) = \exp \left( \int_{t_0}^{t} A(s) ds \right) \]
\[ = \exp \left( - \int_{t_0}^{t} f(s) ds \right) \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix}, \]
for all \( t \in \mathbb{R}_+ \).

Now, if \( z := (x, y)^T \) is a vector of \( \mathbb{R}^2 \) we set \( \|z\| := \sqrt{x^2 + y^2} \).

We first assume that (i), (iii)-(v) are fulfilled. In order to prove that the null solution of (3.1) is stable, take some \( z_0 \neq 0 \) with \( \|z_0\| \) small enough and \( t_0 \geq 0 \) and denote by \( z(t, t_0, z_0) \) the unique solution of (3.1) which is equal to \( z_0 \) for \( t = t_0 \). By our assumptions, we know that \( z(t, t_0, z_0) \) is defined on a maximal right interval, say \( [t_0, b) \). This solution satisfies the integral equation

\[
(3.3) \quad z(t, t_0, z_0) = Z(t, t_0)z_0 + \int_{t_0}^{t} Z(t, t_0)Z(s, t_0)^{-1}[B(s)z(s, t_0, z_0) + F(s, z(s, t_0, z_0))] ds,
\]
for all \( t \in [t_0, b) \). In fact we can show that \( b = +\infty \). If \( a > 0 \) and \( t_0 < a \) then it follows by (3.3) (see [1])

\[
(3.4) \quad \|z(t, t_0, z_0)\| \leq \|z_0\|e^{-\int_{t_0}^{t} f(s) ds} + \int_{t_0}^{t} e^{-\int_{t_0}^{s} f(u) du} [f'(s) + f^2(s) \|z(s, t_0, z_0)\|] ds + M \int_{t_0}^{t} e^{-\int_{t_0}^{s} f(u) du} f(s) \|z(s, t_0, z_0)\|^2 ds,
\]
for all \( t \in [t_0, b) \). Suppose, by contradiction, that \( b < +\infty \). Moreover, in a first stage, assume that \( b \leq a \).

Since \( f \in C^1[0, a] \), it follows by (3.4) that there exists a constant \( D > 0 \) such that

\[
\|z(t, t_0, z_0)\| \leq \|z_0\| + D \int_{t_0}^{t} (\|z(s, t_0, z_0)\| + \|z(s, t_0, z_0)\|^2) ds =: r(t),
\]
for all \( t \in [t_0, b) \),

\[
r(t_0) = \|z_0\|, \quad r(t) \geq \|z_0\| > 0, \quad t \in [t_0, b).
\]

Hence, for \( t \in [t_0, b) \), one gets

\[
(3.5) \quad r'(t) \leq Dr(t) + Dr(t)^2.
\]
By multiplying inequality (3.5) by \((1 - \alpha)\alpha(t)^{-\alpha}\), since \(\alpha > 1\), it follows that
\[
(1 - \alpha)\alpha'(t)\alpha(t)^{-\alpha} \geq (1 - \alpha)\alpha\alpha(t)^{1-\alpha} + (1 - \alpha)\alpha, \quad (\forall) t \in [t_0, b),
\]
or, equivalently,
\[
[(1 + \alpha(t)^{1-\alpha})e^{D(\alpha-1)(t-t_0)}]' \geq 0, \quad (\forall) t \in [t_0, b).
\]
Therefore the mapping \(t \rightarrow (1 + \alpha(t)^{1-\alpha})e^{D(\alpha-1)(t-t_0)}\) is nondecreasing on \([t_0, b)\).

Hence
\[
(1 + \alpha(t)^{1-\alpha})e^{D(\alpha-1)(t-t_0)} \geq 1 + \|z_0\|^{1-\alpha}, \quad (\forall) t \in [t_0, b).
\]

Obviously, if
\[
\|z_0\| < (e^{D(\alpha-1)\alpha} - 1)^{1/(1-\alpha)} =: \delta_1,
\]
it follows from (3.6) that
\[
r(t) \leq ((1 + \|z_0\|^{1-\alpha})e^{D(\alpha-1)\alpha} - 1)^{1/(1-\alpha)}, \quad (\forall) t \in [t_0, b).
\]

Since \(r(t)\) is bounded on \([t_0, b)\), it follows that \(z(t, t_0, z_0)\) and \(z'(t, t_0, z_0)\) are both bounded on \([t_0, b)\) and therefore \(z(t, t_0, z_0)\) can be extended to the right of \(b\). This fact contradicts the maximality of \(b\). Hence, \(z(t, t_0, z_0)\) does exist on \([t_0, b)\) with \(b > a\). Let us still assume that \(b\) is finite, i.e. \(a < b < +\infty\). We are going to establish an estimate for \(z(t, t_0, z_0)\) on the interval \([a, b)\). This time, our assumption (iii) comes into play. Indeed, starting from (3.3), where \(t_0\) and \(z_0\) are replaced by \(a\) and \(z(a, t_0, z_0)\), we get
\[
\|z(t, t_0, z_0)\| \leq \|z(a, t_0, z_0)\|e^{-\int_a^t f(s)ds}
\]
\[+ \int_a^t e^{-\int_a^s f(u)du} |f'(s) + f^2(s)| |x(s, t_0, z_0)| ds
\]
\[+ \int_a^t e^{-\int_a^s f(u)du} |g(s, x(s, t_0, z_0))| ds
\]
\[\leq \|z(a, t_0, z_0)\|e^{-\int_a^t f(s)ds}
\]
\[+ \int_a^t e^{-\int_a^s f(u)du} |Kf'(s)| x(s, t_0, z_0) |\alpha| ds
\]
\[=: u(t), \quad a \leq t < b.
\]
We have used the fact that \( z(t, t_0, z_0) = z(t, a, z(a, t_0, z_0)) \), \( a \leq t < b \). After an easy computation we find

\[
v'(t) = [K f(t) |x(t, t_0, z_0)| + M f(t) |x(t, t_0, z_0)|^2] - f(t) v(t).
\]

Since \( 0 \leq |x(t, t_0, z_0)| \leq \|z(t, t_0, z_0)\| \leq v(t) \), it follows that

\[
v'(t) \leq f(t) [(K - 1) + M v(t)^{x-1}] v(t), \quad t \in [a, b),
\]

\[
v(a) = \|z(a, t_0, z_0)\|.
\]

By multiplying inequality (3.8) by \((1 - x) v(t)^{-x} \), we obtain

\[
(v(t)^{1-x})' \geq (x - 1)(1 - K) f(t) v(t)^{1-x} + M f(t)(1 - x),
\]

for every \( t \in [a, b) \).

By (3.9) we easily deduce the estimate

\[
v(t) \leq \left\{ e^{(x-1)(1-K) \int_{t_0}^t f(s) ds} \left[ \|z(a, t_0, z_0)\|^{1-x} - \frac{M}{1 - K} + \frac{M}{1 - K} \right] \right\}^{1/(1-x)},
\]

for all \( t \in [a, b) \).

If

\[
\|z(a, t_0, z_0)\| \in \left( 0, \left( \frac{1 - K}{M} \right)^{1/(x-1)} \right),
\]

then (3.10) shows that \( z(t, t_0, z_0) \) is bounded on \([a, b)\) and hence \( b = +\infty \).

If \( t_0 \geq a \), then we similarly get

\[
v(t) \leq \left\{ e^{(x-1)(1-K) \int_{t_0}^t f(s) ds} \left[ \|z_0\|^{1-x} - \frac{M}{1 - K} + \frac{M}{1 - K} \right] \right\}^{1/(1-x)},
\]

for all \( t \in [a, b) \).

Again, for

\[
\|z_0\| \in \left( 0, \left( \frac{1 - K}{M} \right)^{1/(x-1)} \right),
\]

\( z(t, t_0, z_0) \) exists on \([t_0, +\infty)\) (i.e., \( b = +\infty \)).

Now, by estimates (3.7) (for \( t \in [t_0, a] \)), (3.10), (3.11), where \( b = +\infty \), we see that the null solution of (1.1) is uniformly stable. If in addition (ii) is fulfilled, then by (3.10), (3.11) (where \( b = +\infty \)), it follows that the null solution of (1.1) is asymptotically stable. The proof of Theorem 2.1 is complete.

\[\square\]

4. Extensions to \( R \)

In this section we present some remarks concerning the possibility to extend Theorem 2.1 to the whole real line \( R \). So, let us consider the equation
\begin{equation}
\tag{4.1}
x'' + 2f(t)x' + x + g(t, x) = 0, \quad t \in \mathbb{R},
\end{equation}

where \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are two given functions, satisfying the following hypotheses:

(i)' \( f \in C^1(\mathbb{R}) \) and \( t \cdot f(t) > 0 \), for all \( t \in \mathbb{R}, \ t \neq 0; \)

(ii)' \( \int_{-\infty}^{0} f(t)dt = -\infty \) and \( \int_{0}^{+\infty} f(t)dt = +\infty; \)

(iii)' there exist \( a \geq 0 \) and \( K \in (0, 1) \) such that

\begin{equation}
\tag{4.2}
|f'(t) + f^2(t)| \leq K|f(t)|, \quad \text{for all } |t| \geq a;
\end{equation}

(iv)' \( g \in C(\mathbb{R} \times \mathbb{R}) \) and \( g \) is locally Lipschitzian in \( x; \)

(v)' there exist \( M > 0 \) and \( \alpha > 1 \) such that

\[ |g(t, x)| \leq M|f(t)| \cdot |x|^\alpha, \quad \text{for all } (t, x) \in \mathbb{R}^2. \]

An example. Let \( f \) be defined by \( f(t) = 1/t \) for \( |t| \geq a \) (\( a > 0 \)) and extended on the interval \((-a, a)\) in such a way that \( f \in C^1(\mathbb{R}) \) and \( t \cdot f(t) \geq 0 \) for all \( t \in \mathbb{R} \) (for instance, if \( a = 1 \) we can choose \( f(t) = t(3 - 2|t|), \ |t| < 1). \)

Also, let \( g \) be defined by \( g(t, x) = f(t) \cdot x^2, \ \alpha > 1 \). It is easily seen that these functions satisfy (i)'–(v)'.

Notice that by the changes

\[ s = -t, \quad u(s) = x(-s), \quad t \leq 0, \]

equation (4.1) for \( t \leq 0 \) becomes

\[ \frac{d^2u}{ds^2} + 2f^*(s) \frac{du}{ds} + u + g^*(s, u) = 0, \quad s \in \mathbb{R}_+, \]

where \( f^*(s) = -f(-s) \) and \( g^*(s, u) = g(-s, u). \)

Taking into account this remark as well as Theorem 2.1 we can state the following result:

**Theorem 4.1.** Suppose that hypotheses (i)', (iii)'–(v)' are fulfilled. Then for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every initial data \( |x(0)| < \delta, \ |x'(0)| < \delta, \) equation (4.1) has a unique solution \( x(t) \) defined on \( \mathbb{R}, \) satisfying \( |x(t)| < \varepsilon, \ |x'(t)| < \varepsilon, \) \( (\forall) t \in \mathbb{R}. \) If, in addition, (ii)' holds, then

\[ x(\pm \infty) = x'(\pm \infty) = 0. \]

**Remark 4.1.** Our extensions to the whole real line \( \mathbb{R} \) are allowed by the fact that the key condition (4.2) is fulfilled away from the origin.

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References


nuna adreso:
Gheorghe Moroșanu
Department of Mathematics and Its
Applications
Central European University
Nador u. 9, H-1051, Budapest
Hungary
E-mail: morosanug@ceu.hu

Cristian Vladimirescu
Department of Mathematics
University of Craiova
13 A.I. Cuza Str., Craiova RO 200585
Romania
E-mail: cvladi@central.ucv.ro

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