A necessary and sufficient condition for input identifiability for linear time-invariant systems

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ABSTRACT

A necessary and sufficient condition for input identifiability for linear autonomous systems is given. The result is based on a finite iterative process and its proof relies on elementary arguments involving matrices, finite dimensional linear spaces, Gronwall’s lemma, and linear differential systems. Our condition is equivalent to the classical condition involving the geometrical concept of controlled invariant [V. Basile, G. Marro, Controlled and Conditioned Invariants in Linear System Theory, Prentice Hall, Englewood Cliffs, NJ, 1992, p. 237] and the dimension reduction algorithm that we propose seems to be useful in designing deconvolution methods.

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1. Introduction

Let $M_{p\times q}(\mathbb{R})$ denote the set of all $p \times q$ matrices with real entries. Consider in a given finite interval $0 \leq t \leq T$ the following linear time-invariant system:

\begin{align*}
    \dot{x} &= Ax + Bu, \\
y &= Hx,
\end{align*}

where $A \in M_{n \times n}(\mathbb{R})$, $B \in M_{n \times d}(\mathbb{R})$, $H \in M_{m \times n}(\mathbb{R})$. $u = u(t)$ is a control policy (input) taking values from $\mathbb{R}^d$, $x = x(t) \in \mathbb{R}^n$ denotes the state of the system, and $y = y(t) \in \mathbb{R}^m$ is the output trajectory.

By the variation of constants formula we can see that for every initial state $x_0 \in \mathbb{R}^n$ and control (input) $u \in L^1(0, T; \mathbb{R}^d)$ the corresponding output is given by

\begin{equation}
y(t) = He^{At}x_0 + \int_0^t He^{A(t-s)}Bu(s)\,ds, \quad \forall t \in [0, T].
\end{equation}

Let $\mathcal{A}C([0, T]; \mathbb{R}^m)$ denote the space of absolutely continuous functions on $[0, T]$ with values in $\mathbb{R}^m$. Let $Q : L^1(0, T; \mathbb{R}^d) \to \mathcal{A}C([0, T]; \mathbb{R}^m)$ be the operator defined by

\begin{equation}
(Qu)(t) = \int_0^t He^{A(t-s)}Bu(s)\,ds, \quad \forall t \in [0, T].
\end{equation}

Obviously, the range of $Q$, denoted $\text{Range } Q$, does not cover the whole $\mathcal{A}C([0, T]; \mathbb{R}^m)$ (in particular, every function from Range $Q$ vanishes at $t = 0$).

We continue with the following definition related to system (1) and (2) [see, e.g., [1, p. 167]]:

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Definition. The system input is said to be identifiable (detectable) if for every initial state $x_0$ and output $y = y(t)$ the corresponding input $u = u(t)$ is unique ($u$ is supposed to exist as long as an output $y$ is produced).

Remark 1. If $u$ is unique for some $x_0$, $y$, then the same property holds for all inputs $u$. In fact, input identifiability (or detectability) for system (1) and (2) means that the kernel of $Q$ is the null space: \( \ker Q = \{0\} \). If the system input is identifiable, then the system is said to be left invertible or ideally observable in the Russian literature (see [2]).

Remark 2. If system (1) and (2) is left invertible (i.e., equivalently, ker $Q = \{0\}$), then the following rank condition holds (see [1, p. 168]):

\[
\text{Rank}(B^T A^T, B^T A^T H^T, \ldots , B^T (A^T)^n H^T) = d,
\]

where the superscript $T$ denotes the matrix transpose. The converse implication is not true (i.e., the above rank condition is not sufficient for left invertibility), as the following simple counterexample shows: $A = \text{the matrix with rows } (-1, 0, 0), (0, 0, 0), (0, 1, 0), (0, 1), (0, 0), H = (1, 1, 0)$. Obviously, the rank condition is satisfied, but ker $Q$ contains the nonzero function $u(t) = \cos(t, -1 + e^{-t})$. Therefore, Theorem 5.5.2 in [1, p. 167] is false.

2. The main result

In the previous section we have seen that the problem of left invertibility for system (1) and (2) reduces to the condition \( Q = \{0\} \). In this section we formulate a necessary and sufficient condition on matrices $A$, $B$, $H$ such that \( Q = \{0\} \).

Our result relies on an iterative process. Namely, we construct iteratively a non-increasing sequence of integers \( \{d_i\} \subset \mathbb{N} \) as well as sequences of matrices \( \{A_i\} \subset M_{n \times n}(\mathbb{R}) \), \( \{B_i\} \subset M_{n \times d_i}(\mathbb{R}) \), and \( \{H_i\} \subset M_{n \times n}(\mathbb{R}) \) for \( i = 0, 1, 2, \ldots \) as follows.

Let \( A_0 = A, B_0 = B, H_0 = H \) and \( d_0 = d \). Given \( A_i, B_i, H_i \) and \( d_i \), let dim ker \( (H_i B_i) \) = \( d_i - 1 \). If \( d_i + 1 = 0 \), the iterations terminate. Let \( d_{i+1} > 0 \). Then, \( \dim \text{Range}(H_i B_i) = d_i - d_{i+1} \). Moreover, if \( \mathbb{R}^{d_i} = U_i + \ker(H_i B_i) \) and \( \mathbb{R}^{d_{i+1}} = V_{i+1} \oplus \text{Range}(H_i B_i) \), then \( U_i \cong \ker(H_i B_i) \), \( V_i := \text{Range}(H_i B_i) \), and \( U_i \cong \text{Range}(H_i B_i) \cong \mathbb{R}^{d_i - d_{i+1}} \).

First, assume that \( d_i > d_{i+1} \). Then, there are matrices \( M_i \in M_{d_i \times (d_i - d_{i+1})}(\mathbb{R}) \), \( T_i \in M_{d_i \times d_{i+1}}(\mathbb{R}) \), \( C_i \in M_{(d_i - d_{i+1}) \times m}(\mathbb{R}) \), such that

\[
\text{Range}(M_i) = U_i, \quad \text{Range}(T_i) = \ker(H_i B_i),
\]

\[C_i H_i B_i M_i = \text{id}|_{\mathbb{R}^{d_i - d_{i+1}}} \]

and for any \( x \in \mathbb{R}^{d_i} \) there exist vectors \( y \in \mathbb{R}^{d_i - d_{i+1}}, z \in \mathbb{R}^{d_{i+1}} \), uniquely determined, such that \( x = M_i y + T_i z \). Let \( P_i \) be the matrix of the projection on \( V_i \) with respect to \( \text{Range}(H_i B_i) \). The matrix \( C_i \) may be chosen as

\[
C_i = (M_i^T B_i^T H_i^T H_i B_i M_i)^{-1} M_i^T B_i^T H_i^T.
\]

Indeed, the matrix \( M_i^T B_i^T H_i^T H_i B_i M_i \) is the Gramm matrix of the vectors defined by the columns of \( H_i B_i M_i \) and due to the fact that they are linearly independent \( C_i \) is well-defined. Note that the matrices \( M_i, T_i, C_i \) and \( P_i \) are not uniquely determined.

We define the matrices \( A_{i+1}, B_{i+1} \) and \( H_{i+1} \) by

\[
A_{i+1} := (I - B_i M_i C_i H_i) A_i, \quad B_{i+1} := B_i T_i, \quad H_{i+1} := P_i H_i A_i.
\]

Now, assume that \( d_{i+1} = d_i \). In this case we define

\[
A_{i+1} := A_i, \quad B_{i+1} := B_i, \quad H_{i+1} := H_i A_i.
\]

Let us now state our main result:

**Theorem 1.** Let \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times d} \), \( H \in \mathbb{R}^{n \times n} \), and let \( Q \) be the operator defined before by (4). If \( \{d_i\} \subset \mathbb{N} \), \( \{A_i\} \subset \mathbb{R}^{n \times n} \), \( \{B_i\} \subset \mathbb{R}^{n \times d_i} \) and \( \{H_i\} \subset \mathbb{R}^{n \times n} \) are the sequences defined above, then ker \( Q = \{0\} \) if and only if there exists \( i \in \mathbb{N}, i \leq \sum d_i \) such that \( d_i = 0 \). Moreover, if \( d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_{i+1} > 0 \) for some \( i \), then \( d_k = d_{k-1} \) for all \( k \geq i + 1 \).

**Proof.** Let \( w \in \ker Q \), i.e., \( w \in L^1(0, T; \mathbb{R}^d) \) and

\[
\int_0^T H e^{A(t-s)} B w (s) ds = 0 \quad \forall t \in [0, T].
\]

By differentiation with respect to \( t \) we see that for a.a. \( t \in [0, T] \)

\[
H B w (t) + \int_0^t H A e^{A(t-s)} B w (s) ds = 0.
\]

First, assume that \( d_1 = \dim \ker (H B) < d = d_0 \). If \( d_1 = 0 \), then there exists a matrix \( C \) such that \( CHB = \text{id}_{|\mathbb{R}^d} \) which allows us to write

\[
w (t) + C \int_0^t H A e^{A(t-s)} B w (s) ds = 0.
\]
Now, Gronwall’s lemma implies that \( w \) is the null function. Let \( d_1 > 0 \) and \( w(t) = M_0v(t) + T_0w_1(t) \). Then Eq. (6) is equivalent to the system

\[
v(t) + C \int_0^t H_0A_0e^{A_0(t-s)} B_0(M_0v(s) + T_0w_1(s)) \, ds = 0, \tag{8}
\]

\[
P_0 \int_0^t H_0A_0e^{A_0(t-s)} B_0(M_0v(s) + T_0w_1(s)) \, ds = 0. \tag{9}
\]

It follows by standard arguments that for each integrable function \( w_1(t) \) there is a unique solution \( v(t) \) of Eq. (8). Moreover, \( v \) can be expressed as a convolution product of a suitable matrix kernel and \( w_1 \) (see (12)). Indeed, let us define

\[
V(t) := \int_0^t e^{A_0(t-s)} B_0(M_0v(s) + T_0w_1(s)) \, ds. \tag{10}
\]

We have for a.a. \( t \in [0, T] \)

\[
v(t) = A_0V(t) + B_0M_0v(t) + B_0T_0w_1(t),
\]

which implies (see (8))

\[
V(t) = (A_0 - B_0M_0C_0H_0A_0)V(t) + B_0T_0w_1(t).
\]

Since \( V(0) \) is the null matrix, we obtain by the variation of constants formula

\[
V(t) = \int_0^t e^{(A_0 - B_0M_0C_0H_0A_0)(t-s)} B_0T_0w_1(s) \, ds. \tag{11}
\]

Thus

\[
v(t) = -C_0H_0A_0 \int_0^t e^{(A_0 - B_0M_0C_0H_0A_0)(t-s)} B_0T_0w_1(s) \, ds \tag{12}
\]

is the (unique) solution of Eq. (8) corresponding to \( w_1 \). For this \( v \), according to (10) and (11), we can write

\[
P_0 \int_0^t H_0A_0e^{A_0(t-s)} B_0(M_0v(s) + T_0w_1(s)) \, ds = P_0H_0A_0 \int_0^t e^{(A_0 - B_0M_0C_0H_0A_0)(t-s)} B_0T_0w_1(s) \, ds.
\]

Thus it is obvious that the existence of a nonzero solution \( w(t) \) of Eq. (5) (which is equivalent to system (8) and (9)) is equivalent to the existence of a nonzero solution \( w_1(t) \) of the equation

\[
\int_0^t H_1 e^{A_1(t-s)} B_1 w_1(s) \, ds = 0, \quad w_1(t) \in \mathbb{R}^{d_1},
\]

where

\[
H_1 := P_0H_0A_0, \quad A_1 := (I - B_0M_0C_0H_0)A_0, \quad B_1 := B_0T_0.
\]

Next, assume that \( d_1 = \dim \ker (HB) = d = d_0 \). Then Eq. (6) reads

\[
\int_0^t HA e^{A(t-s)} B w(s) \, ds = 0, \tag{13}
\]

so we can take \( u_1 = w_1 \),

\[
A_1 := A_0, \quad B_1 := B_0, \quad H_1 := H_0A_0,
\]

and we can continue further to construct iteratively matrices \( A_i, B_i \) and \( H_i \). Notice that if for some integer \( i \) we have \( d_i = d_{i+1} = \cdots = d_{i+n-1} \), then

\[
H_i B_i = H_i A_i B_i = \cdots = H_i A_0^{i-1} B_i = \mathcal{O},
\]

where \( \mathcal{O} \) denotes the zero matrix in \( M_{m \times d_i}(\mathbb{R}) \). Thus the Cayley–Hamilton theorem implies that \( H_i A_i^{k} B_i = \mathcal{O} \) for any integer \( k \geq 1 \). In particular, it follows that \( H_i e^{A_0^{k} t} B_i = \mathcal{O} \) and there exists a nonzero function \( u(t) \) satisfying (5). Moreover, either there exists an integer \( i \leq d_n \), such that \( d_i = 0 \) and the iterations terminate or \( 0 < d_{dn} = d_{dn+1} = \cdots = d_{dn+k} = \cdots \) and there exists a nonzero solution of Eq. (5). □
3. Concluding comments

First of all, it has been brought to our attention that our necessary and sufficient condition formulated in Theorem 1 is a variation of the classical condition described in Property 4.3.6 of Basile and Marro [2, Chapter 4], which involves the geometrical concept of controlled invariant. Indeed, our condition is equivalent to the classical one. This can be shown by comparing the two corresponding algorithms, under the maximal rank condition on $B$. We do not assume in Theorem 1 that $B$ has maximal rank (i.e., equivalently, ker $B = \{0\}$), but obviously it is a necessary condition for left invertibility. This equivalence confirms the validity of our result.

While the classical approach is geometrical, our new iterative process relies on simple arguments from linear algebra and the theory of differential equations which allow dimension reduction, as described in the proof of Theorem 1. If the system is left invertible (i.e., ker $Q = \{0\}$), one can use an iterative process suggested by the proof of Theorem 1 to solve for $u = u(t)$ the equation $Qu(t) = f(t)$. This operation is nowadays called *deconvolution* since $Q$ is an integral convolution operator. In general $u$ does not depend continuously on $f$ and this makes the problem difficult. Among the existing papers addressing deconvolution methods, we refer the reader to [3–5] and the references therein. We think that our iterative process could generate new efficient deconvolution methods.

The general output equation $y(t) = Hx(t) + Du(t), \ t \in [0, T]$, can also be considered in our framework. Here $D$ is an $m \times d$ matrix with real entries. In this case, instead of Eq. (5), we have the following integral equation:

$$Dw(t) + H \int_0^t e^{A(t-s)}Bw(s) \, ds = 0,$$

whose form is similar to Eq. (6). Therefore, one can apply our algorithm described in the proof of Theorem 1 above to derive a necessary and sufficient condition for input identification (or left invertibility). The precise formulation of this condition is left to the reader. In particular, if ker $D = \{0\}$, then obviously the system input is identifiable.

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References