MULTIPLICITY RESULTS FOR DOUBLE EIGENVALUE PROBLEMS INVOLVING THE $p$-LAPLACIAN

Hannelore Lisei$^1$, Gheorghe Moroşanu and Csaba Varga$^2$

Abstract. The existence of multiple nontrivial solutions for two types of double eigenvalue problems involving the $p$-Laplacian is derived. To prove the existence of at least two nontrivial solutions we use a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12]. The existence of at least three nontrivial solutions is shown by combining a result of B. Ricceri [17] and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12].

1. INTRODUCTION

Let $h_p : \mathbb{R}^N \to \mathbb{R}^N$ be the homeomorphism defined by $h_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}^N$, where $p > 1$ is fixed and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

For $T > 0$, let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a mapping satisfying:

$(F_1)$ for each $M > 0$ there exists some $\alpha_M \in L^1(0, T)$ such that, for a.e. $t \in [0, T]$ and all $x, y \in B_M = \{\xi \in \mathbb{R}^N : |\xi| \leq M\}$, it holds

$$|F(t, x) - F(t, y)| \leq \alpha_M(t)|x - y|;$$

$(F_2)$ the mapping $F(\cdot, x) : [0, T] \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}^N$ and $F(\cdot, 0) \in L^1(0, T)$;

$(F_3)$ $\lim_{|x| \to \infty} \frac{F(t, x) - F(t, 0)}{|x|^p} \leq 0$ uniformly for a.e. $t \in [0, T]$.

Let $j : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty, +\infty]$ be a function having the following properties:

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\(D(j) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : j(x, y) < +\infty \} \neq \emptyset\) is a closed convex cone with \(D(j) \neq \{(0, 0)\};\)

\((J_2)\) \(j\) is a convex and lower semicontinuous (shortly, l.s.c.) function.

Let \(\gamma > 0\) be arbitrary. For \(\lambda, \mu > 0\) we consider the following double eigenvalue problem involving the \(p\)-Laplacian operator:

\[
(P_{\lambda, \mu}) \quad \begin{cases}
- [h_p(u')'] + \gamma h_p(u) & \in \lambda \partial F(t, u) \ \text{a.e.} \ t \in [0, T], \\
(h_p(u')(0), -h_p(u')(T)) & \in \mu \partial j(u(0), u(T)),
\end{cases}
\]

where \(u : [0, T] \to \mathbb{R}^N\) is of class \(C^1\) and \(h_p(u')\) is absolutely continuous. Note, that \(\partial F(t, \eta)\) denotes the generalized gradient (in the sense of Clarke) of \(F(t, \cdot)\) at \(\eta \in \mathbb{R}^N\), while \(\partial j\) denotes the subdifferential of \(j\) in the sense of convex analysis.

Our approach to problem \((P_{\lambda, \mu})\) is a variational one and it relies on results concerning Motreanu-Panagiotopoulos type functionals (see for example in [13] and [14]), which are extensions of the critical point theory of Szulkin type functionals [18].

Previous results concerning \(p\)-Laplacian systems with various types of boundary conditions have been obtained by R. Manásevich and J. Mawhin [8], [9], J. Mawhin [10], [11], L. Gasinski and N. Papageorgiu [4], P. Jebelean and G. Moroşanu [6], [7]. As far as we know, eigenvalue problems for differential inclusions involving the \(p\)-Laplacian and having mixed boundary conditions where not studied yet. Eigenvalue problems with no boundary conditions were investigated in the books [13],[14] (see also the references therein).

In order to obtain the existence of multiple solutions for problem \((P_{\lambda, \mu})\) we impose some further assumptions on \(F\):

\((F_4)\) \(\lim_{|x| \to 0} \frac{F(t, x) - F(t, 0)}{|x|^p} \leq 0\) uniformly for a.e. \(t \in [0, T]\);

\((F_5)\) there exists \(s_0 \in \mathbb{R}^N\) such that \(\int_0^T (F(t, s_0) - F(t, 0)) dt > 0\).

P. Jebelean and G. Moroşanu [6] proved the existence of a nontrivial solution for a differential inclusion problem of the type \((P_{\lambda, \mu})\) by using “mountain pass theorems”. Our paper completes their results by proving the existence of at least two nontrivial solutions for a first type of double eigenvalue problem and the existence of at least three nontrivial solutions for a second type of double eigenvalue problem. For this, we need assumptions on the behavior around zero and close to infinity of the function \(F\) (see \((F_3), (F_4), (F_5))\). The two types of problems \((P_{\lambda, \mu})\) rely on different assumptions for the function \(j\), and for this reason we use different tools for their investigation.
The main tool for the first type problem is a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12, Theorem 3.1]. For the second type problem we use a recent result of B. Ricceri [17, Theorem 4] concerning the existence of multiple solutions and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12, Corollary 2.1].

This paper is organized as follows: in Section 2, there are introduced some notations and important preliminary results for problem \((P_{\lambda,\mu})\). Then, in Section 3 it is proved the existence of at least two nontrivial solutions for the first type double eigenvalue problem \((P_{\lambda,\mu})\) and in Section 4 we complete the results of Section 3 by showing the existence of at least three nontrivial solutions for the second type double eigenvalue problem \((P_{\lambda,\mu})\). Finally, Section 5 contains important results from variational calculus concerning the critical point theory, which are used in our investigations.

2. NOTATIONS AND PRELIMINARY RESULTS

Let \(W^{1,p} = W^{1,p}(0, T; \mathbb{R}^N)\) be the usual Sobolev space equipped with the norm

\[\|u\|_\eta = \left(\|u'\|_{L^p}^p + \eta\|u\|_{L^p}^p\right)^{1/p},\]

where \(\eta > 0\), and \(\| \cdot \|_{L^p}\) is the norm of \(L^p = L^p(0, T; \mathbb{R}^N)\)

\[\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}.\]

We consider \(C = C([0, T]; \mathbb{R}^N)\) endowed with the norm

\[\|u\|_C = \max\{|u(t)| : t \in [0, T]\}.\]

For \(\gamma > 0\), we consider \(\varphi_\gamma : W^{1,p} \to \mathbb{R}\) defined by

\[\varphi_\gamma(u) := \frac{1}{p}\left(\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p\right)\text{ for all } u \in W^{1,p}.\]

Note, that \(\varphi_\gamma\) is convex and \(\varphi_\gamma \in C^1(W^{1,p}; \mathbb{R})\) with

\[\langle \varphi_\gamma'(u), v \rangle = \int_0^T (h_p(u'), v')dt + \gamma \int_0^T (h_p(u), v)dt\text{ for all } u, v \in W^{1,p}.\]

We define the function \(J : W^{1,p} \to ]-\infty, +\infty]\) by

\[J(u) = j(u(0), u(T))\text{ for all } u \in W^{1,p}.\]
$J$ is a proper, convex and l.s.c. function. Note, that
\[ D(J) = \{ u \in W^{1,p} : (u(0), u(T)) \in D(j) \}. \]
We introduce the constant $\gamma_1 = \gamma_1(p, \gamma) > 0$ by setting
\[ \gamma_1 = \inf \left\{ \frac{\|u'\|^p_{L^p} + \gamma \|u\|^p_{L^p}}{\|u\|^p_{L^p}} : u \in W^{1,p} \setminus \{0\}, u \in D(J) \right\}. \]
By computation one has
\[ 2^{-1/p} \|u\|_{\gamma_1} \leq (\|u'\|^p_{L^p} + \gamma \|u\|^p_{L^p})^{1/p} \leq \|u\|_{\gamma_1} \text{ for all } u \in D(J). \quad (2.1) \]
We consider the functional $\hat{\mathcal{F}} : C \to \mathbb{R}$ defined by
\[ \hat{\mathcal{F}}(v) = -\int_0^T F(t, v)dt + \int_0^T F(t, 0)dt \text{ for all } v \in C \]
and $\mathcal{F} : W^{1,p} \to \mathbb{R}$ defined by $\mathcal{F} = \hat{\mathcal{F}}|_{W^{1,p}}$. The functional $\mathcal{F}$ is sequentially weakly continuous, since the embedding $W^{1,p} \hookrightarrow C$ is compact.

Note that for $1 \leq r < p$ and $p < q < p^*$ the embeddings $L^p \hookrightarrow L^r$, $W^{1,p} \hookrightarrow L^q$, $W^{1,p} \hookrightarrow C$ are continuous, hence there exist constants $C_{r,p}, \hat{C}_{q,p}, \hat{c} > 0$ such that
\[ \|u\|_{L^r} \leq C_{r,p} \|u\|_{L^p}, \quad \|u\|_{L^q} \leq \hat{C}_{q,p} \|u\|_{W^{1,p}}, \quad \|u\|_C \leq \hat{c} \|u\|_{W^{1,p}} \text{ for all } u \in W^{1,p}. \]

Let $\mathcal{E} : [0, \infty) \times [0, \infty) \times W^{1,p} \to ]-\infty, +\infty [$ be defined by
\[ \mathcal{E}(\lambda, \mu, u) = \varphi_\gamma(u) + \lambda \mathcal{F}(u) + \mu J(u). \]
The functional $\mathcal{E}$ is of Motreanu-Panagiotopoulos type.

**Proposition 2.1.** [6, Proposition 3.2]. Assume that $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (F1) and (F2) and $j : \mathbb{R}^N \times \mathbb{R}^N \to ]-\infty, +\infty [$ satisfies (J1) and (J2). If $u \in W^{1,p}$ is a critical point of $\mathcal{E}(\lambda, \mu, \cdot)$ (in the sense of Definition 5.1), then $u$ is a solution of (P$\lambda, \mu$).

**Remark 2.1.** Let $\varepsilon > 0$ be arbitrary. From (F1), (F2) and (F3) it follows that there exists $\delta_1 > 0$ (depending on $\varepsilon$) such that
\[ F(t, x) - F(t, 0) \leq \varepsilon |x|^p + \alpha_{\delta_1}(t) \delta_1 \quad \text{for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0, T]. \]
Then
\[ \mathcal{F}(u) \geq -\varepsilon \|u\|_{L^p}^p - \delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} \quad \text{for all } u \in W^{1,p}. \quad (2.2) \]
Proposition 2.2. Assume that $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies $(F_1)$, $(F_2)$ and $(F_3)$ and that $j : \mathbb{R}^N \times \mathbb{R}^N \to ] - \infty, +\infty]$ satisfies $(J_1)$ and $(J_2)$. Then the following properties hold:

1. $\mathcal{E}(\lambda, \mu, \cdot)$ is weakly sequentially lower semicontinuous on $W^{1,p}$ for each $\lambda > 0, \mu \geq 0$;
2. $\lim_{\|u\|_{\gamma_1} \to +\infty} \mathcal{E}(\lambda, \mu, u) = +\infty$ for each $\lambda > 0, \mu \geq 0$;
3. $\mathcal{E}(\lambda, \mu, \cdot)$ satisfies the $(PS)$ condition for each $\lambda, \mu > 0$.

Proof. (1) The function $\mathcal{E}(\lambda, \mu, \cdot)$ is weakly sequentially l.s.c on $W^{1,p}$, because $F$ is weakly sequentially l.s.c., while $\varphi_\gamma$ and $J$ are convex and l.s.c., hence they are also weakly sequentially l.s.c.

(2) First observe that

$$
\|u\|_{L^p}^p \leq \frac{1}{\gamma_1} \|u\|_{\gamma_1}^p \text{ for all } u \in W^{1,p}.
$$

In (2.2) we choose $\varepsilon < \frac{2\lambda}{\gamma_1 p}$. Using that the embedding $L^p \hookrightarrow L^1$ is continuous and that (2.1) holds, we have for all $u \in D(J)$

$$
\mathcal{E}(\lambda, \mu, u) \geq \frac{1}{p} \left( \|u\|_{L^p}^p + \gamma \|u\|_{L^p}^p \right) - \lambda \varepsilon \|u\|_{L^p}^p - \lambda \delta_1 \|\alpha \delta_1\|_{L^1(0,T)} + \mu J(u)
$$

$$
\geq \frac{\gamma_1 - 2\varepsilon \lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda \delta_1 \|\alpha \delta_1\|_{L^1(0,T)} + \mu J(u).
$$

Since $J$ is convex and l.s.c. it is bounded from below by an affine functional and then there exist constants $c_1, c_2, c_3 > 0$ such that for all $u \in D(J)$

$$
\mathcal{E}(\lambda, \mu, u) \geq \frac{\gamma_1 - 2\varepsilon \lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda \delta_1 \|\alpha \delta_1\|_{L^1(0,T)} - c_1 |u(0)| - c_2 |u(T)| - c_3.
$$

By the continuity of the embedding $W^{1,p} \hookrightarrow C$ we have for all $u \in W^{1,p}$

$$
\mathcal{E}(\lambda, \mu, u) \geq c_4 \|u\|_{\gamma_1}^p - c_5 \|u\|_{\gamma_1}^p - c_6,
$$

where $c_4, c_5, c_6 > 0$ are constants. Since, $1 < p$ it follows that $\mathcal{E}(\lambda, \mu, \cdot) \to +\infty$ when $\|u\|_{\gamma_1} \to +\infty$.

(3) Let $\{u_n\}$ in $W^{1,p}$ be a sequence satisfying $\mathcal{E}(\lambda, \mu, u_n) \to c$ and

$$
\lambda \varphi_0(u_n; v - u_n) + \varphi_\gamma(v) - \varphi_\gamma(u_n) + \mu J(v) - \mu J(u_n) \geq -\varepsilon_n \|v - u_n\|_{\gamma_1}, \forall v \in W^{1,p},
$$

where $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$. We have a subsequence $\{u_n\} \subset D(J)$ (we just eliminate the finite number of elements of the sequence which do not belong to $D(J)$), since $\mu > 0$ and $\mathcal{E}(\lambda, \mu, u_n) \to c$. 
But \( E(\lambda, \mu, \cdot) \) is coercive, this implies that \( \{u_n\} \) is bounded in \( W^{1,p} \). The embedding \( W^{1,p} \hookrightarrow C \) is compact, then we can find a subsequence, which we still denote by \( \{u_n\} \), which is weakly convergent to a point \( u \in W^{1,p} \) and strongly in \( C \).

In the above inequality we take \( v = u_n + s(u - u_n) \), with \( s > 0 \), then divide both sides of the inequality by \( s \) and let \( s \downarrow 0 \), to obtain

\[
\lambda \Phi^0(u_n; u - u_n) + \phi'(u_n; u - u_n) + \mu J'(u_n; u - u_n) \geq -\varepsilon_n \|u - u_n\|_{\gamma_1}, \quad \forall n \in \mathbb{N}.
\]

By the upper semicontinuity of \( \Phi^0 \) (see [14], Chapter 1), it follows that

\[
\liminf_{n \to \infty} \left( \phi'(u_n; u - u_n) + \mu J'(u_n; u - u_n) \right) \geq 0.
\]

By Lemma 4.1 in [6] it follows that \( \{u_n\} \) converges strongly to \( u \in W^{1,p} \). \( \blacksquare \)

**Remark 2.2.** From \( (F_1), (F_2), (F_3) \) and \( (F_4) \) it follows that for each \( \varepsilon > 0 \) there exist \( \delta_\varepsilon, \bar{\delta}_\varepsilon > 0 \) such that

\[
F(t, x) - F(t, 0) \leq \varepsilon |x|^p + \frac{\alpha_\delta(t)}{\delta_\varepsilon} |x|^r \quad \text{for all} \ x \in \mathbb{R}^N, \ \text{a.e.} \ t \in [0, T],
\]

where \( r \geq 1 \). Then, by using the continuity of the embedding \( W^{1,p} \hookrightarrow C \) we get

\[
\Phi(u) \geq -\varepsilon \|u\|_{L^p}^p - \frac{\partial_i^m \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\delta_\varepsilon^{p-1}} \|u\|_{\gamma}^r \quad \text{for all} \ u \in W^{1,p}.
\]

**Remark 2.3.** If \( F : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) satisfies \( (F_1) \) and \( (F_4) \), then \( 0 \in \bar{\partial}F(t, 0) \) for a.e. \( t \in [0, T] \). In order to prove this property, let \( x \in \mathbb{R}^N \) be fixed. From \( (F_4) \) it follows that there exists \( \delta > 0 \) such that

\[
F(t, z) - F(t, 0) \leq |z|^p \quad \text{for each} \ |z| < \delta \quad \text{and a.e.} \ t \in [0, T].
\]

But

\[
(-F)^0(t, 0; x) = \lim_{\varepsilon \downarrow 0} \sup_{0 < |w| < \varepsilon} \frac{-F(t, w + sx) + F(t, w)}{s}.
\]

Let \( \varepsilon > 0 \) be fixed and let \( \{w_n\} \) be a sequence in \( \mathbb{R}^N \) such that \( |w_n| \searrow 0 \) and \( |w_n| < \varepsilon \) for all \( n \in \mathbb{N} \). Then for \( 0 < s < \varepsilon \) and \( n \in \mathbb{N} \) we have

\[
\frac{-F(t, w_n + sx) + F(t, w_n)}{s} \leq \sup_{0 < |w| < \varepsilon} \frac{-F(t, w + sx) + F(t, w)}{s}.
\]
Since $F(t, \cdot)$ is continuous (see $(F_1)$), we get for $n \to \infty$

$$-F(t, sx) + F(t, 0) \leq \sup_{0 < |w| < \varepsilon, 0 < s \leq \varepsilon} \frac{-F(t, w + sx) + F(t, w)}{s},$$

when $0 < s < \varepsilon$. By (2.4) it follows that

$$-s^{p-1}|x|^p \leq \sup_{0 < |w| < \varepsilon, 0 < s \leq \varepsilon} \frac{-F(t, w + sx) + F(t, w)}{s},$$

when $s$ is small enough such that $|sx| < \delta$. Finally we take $\varepsilon \to 0$ and get

$$0 \leq (-F)^0(t, 0; x) = F^0(t, 0; -x) \text{ for all } x \in \mathbb{R}^N.$$ This implies, $0 \in \partial F(t, 0)$ for a.e. $t \in [0, T]$.

3. **First Type Problem**

In order to obtain the existence of at least two nontrivial solutions for $(P_{\lambda, \mu})$ we impose some further assumptions on the convex function $j : \mathbb{R}^N \times \mathbb{R}^N \to ]-\infty, +\infty]$ which satisfies $(J_1)$ and $(J_2)$:

$(J_3)$ $j(0, 0) = 0$, $j(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

**Theorem 3.1.** Let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1)-(F_5)$ and let $j : \mathbb{R}^N \times \mathbb{R}^N \to ]-\infty, +\infty]$ be a function satisfying $(J_1)-(J_3)$. Then for each fixed $\mu > 0$, there exists an open interval $\Lambda, \mu \subset ]0, +\infty[$ such that for each $\lambda \in \Lambda, \mu$, the problem $(P_{\lambda, \mu})$ has at least two nontrivial solutions.

**Proof.** Let $\mu > 0$ be fixed. We define the function $g : ]0, +\infty[ \to \mathbb{R}$, by

$$g(t) = \sup \{-F(u) : \varphi_\gamma(u) + \mu J(u) \leq t\}, \text{ for all } t > 0.$$ Using (2.3) for $r \in [p, p^*]$ it follows that for all $u \in W^{1, p}$ we have

$$-F(u) \leq \frac{\varepsilon}{\gamma} \|u\|_{p}^p + \frac{\alpha \gamma}{\delta^{n-1}} \|L^1(0, T)\| \|u\|_{\gamma}^p.$$ Since $p < r$, this implies

$$\lim_{t \to 0^+} \frac{g(t)}{t} = 0.$$ Using $(F_5)$ we define $u_0(t) = s_0$ for a.e. $t \in [0, T]$. Then, $u_0 \in W^{1, p} \setminus \{0\}$ and $-F(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number $t_0$ such that $0 < t_0 < \varphi_\gamma(u_0) + \mu J(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-F(u_0)).$$
We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < \left[\varphi_\gamma(u_0) + \mu J(u_0)\right]^{-1} \cdot (-F(u_0))t_0.$$  

(3.1)

We apply Theorem 5.2 to the space $W^{1,p}$, the interval $\Lambda = [0, +\infty[$ and the functions $\mathcal{G}, \mathcal{H} : W^{1,p} \to \mathbb{R}$, $h : \Lambda \to \mathbb{R}$ defined by

$$\mathcal{G}(u) = \varphi_\gamma(u), \psi(u) = \mu J(u), \mathcal{H}(u) = F(u), h(\lambda) = \rho_0 \lambda.$$  

By Proposition 2.2 the assumption (a) from Theorem 5.2 is fulfilled.

We prove now the minimax inequality

$$\sup_{\lambda \in \Lambda} \inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right) < \inf_{u \in W^{1,p}} \sup_{\lambda \in \Lambda} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right).$$

The function

$$\lambda \mapsto \inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right)$$

is upper semicontinuous on $\Lambda$. Since

$$\inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right) \leq \varphi_\gamma(u_0) + \mu J(u_0) + \lambda F(u_0) + \rho_0 \lambda$$

and $\rho_0 < -F(u_0)$, it follows that

$$\lim_{\lambda \to +\infty} \inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right) = -\infty.$$  

Thus we can find $\overline{\lambda} \in \Lambda$ such that

$$\beta_1 := \sup_{\lambda \in \Lambda} \inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \lambda F(u) + \rho_0 \lambda \right)$$

$$= \inf_{u \in W^{1,p}} \left( \varphi_\gamma(u) + \mu J(u) + \overline{\lambda} F(u) + \rho_0 \overline{\lambda} \right).$$

In order to prove that $\beta_1 < t_0$, we distinguish two cases:

I. If $0 \leq \overline{\lambda} < \frac{t_0}{\rho_0}$, we have

$$\beta_1 \leq \varphi_\gamma(0) + \mu J(0) + \overline{\lambda} F(0) + \rho_0 \overline{\lambda} = \overline{\lambda} \rho_0 < t_0.$$  

II. If $\overline{\lambda} \geq \frac{t_0}{\rho_0}$, then we use $\rho_0 < -F(u_0)$ and the inequality (3.1) to get

$$\eta_1 \leq \varphi_\gamma(u_0) + \mu J(u_0) + \overline{\lambda} F(u_0) + \rho_0 \overline{\lambda} \leq \varphi_\gamma(u_0) + \mu J(u_0) + \frac{t_0}{\rho_0} (\rho_0 + F(u_0)) < t_0.$$
From \( g(t_0) < \rho_0 \) it follows that for all \( u \in W^{1,p} \) with \( \varphi_{\gamma}(u) + \mu J(u) \leq t_0 \) we have \( -\mathcal{F}(u) < \rho_0 \). Hence

\[
  t_0 \leq \inf \{ \varphi_{\gamma}(u) + \mu J(u) : -\mathcal{F}(u) \geq \rho_0 \}.
\]

On the other hand,

\[
  \beta_2 = \inf_{u \in W^{1,p}} \sup_{\lambda \in \Lambda} (\varphi_{\gamma}(u) + \mu J(u) + \lambda \mathcal{F}(u) + \rho_0 \lambda) = \inf \{ \varphi_{\gamma}(u) + \mu J(u) : -\mathcal{F}(u) \geq \rho_0 \}.
\]

We conclude that \( \beta_1 < t_0 \leq \beta_2 \).

Hence, assumption (b) from Theorem 5.2 holds. Then, by Theorem 5.2 it follows that there exists an open interval \( \Lambda_\mu \subseteq [0, \infty) \) such that for each \( \lambda \in \Lambda_\mu \) the function \( \varphi_{\gamma} + \mu J + \lambda \mathcal{F} \) has at least three critical points in \( W^{1,p} \). By Proposition 2.1 it follows that these critical points are solutions of \((P_{\lambda,\mu})\). Since \( 0 \in \partial \mathcal{F}(t,0) \) for a.e. \( t \in [0,T] \), we get that at least two of the above solutions are nontrivial.

**Remark 3.1.** The two conditions from \((J_3)\) can be replaced by

\[
  (J_3') \quad j(x, y) \geq j(0, 0) \quad \text{for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.
\]

Then, all the proofs above can be adapted by considering

\[
  J(u) = j(u(0), u(T)) - j(0, 0).
\]

**Corollary 3.1.** Let \( F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a function satisfying \((F_1) - (F_5)\) and let \( b : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a positive, convex and Gâteaux differentiable function with \( b(0,0) = 0 \). Assume that \( S \subset \mathbb{R}^N \times \mathbb{R}^N \) is a nonempty closed convex cone with \( S \neq \{(0,0)\} \), whose normal cone we denote by \( N_S \). Then for each fixed \( \gamma, \mu > 0 \), there exists an open interval \( \Lambda_0 \subseteq [0, +\infty] \) such that for each \( \lambda \in \Lambda_0 \), the following problem

\[
  (\hat{P}_{\lambda,\mu}) \begin{cases} 
  -[h_p(u')]' + \gamma h_p(u) \in \lambda \partial \mathcal{F}(t, u) \text{ a.e. } t \in [0,T], \\
  (u(0), u(T)) \in S, \\
  \left( h_p(u')(0), -h_p(u')(T) \right) \in \mu \nabla b(u(0), u(T)) + \mu N_S(u(0), u(T)),
  \end{cases}
\]

has at least two nontrivial solutions.

**Proof.** The statement follows by applying Theorem 3.1 to the function \( F \) and the convex function \( j : \mathbb{R}^N \times \mathbb{R}^N \rightarrow ]-\infty, +\infty[ \) defined by

\[
  j(x, y) = b(x, y) + I_S(x, y), \quad \text{for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
where

\[ I_S(x, y) = \begin{cases} 
0, & \text{if } (x, y) \in S \\
+\infty, & \text{if } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus S,
\end{cases} \]

is the indicator function of the cone \( S \).

Note, that in this case \( D(j) = S \) and \( j \) satisfies the conditions \((J_1) - (J_3)\).

Moreover,

\[ \partial j(x, y) = \nabla b(x, y) + \partial I_S(x, y) = \nabla b(x, y) + N_S(x, y) \text{ for all } (x, y) \in S. \]

Example 3.1. We give an example of a function \( F \) that satisfies the assumptions \((F_1)\) to \((F_5)\): Let \( F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) be defined by

\[ F(t, x) = f(t) - \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\} \text{ for all } t \in [0, T], x \in \mathbb{R}^N, \]

where \( \alpha > 0, \beta \in [0, p[ \}, f \in L^1(0, T). \)

Various possible choices of \( b \) and \( S \) from Corollary 3.1 recover some classical boundary conditions. For instance:

(a) \( b = 0 \) and \( S = \{(x, x) : x \in \mathbb{R}^N\} \) we get periodic boundary conditions

\[ u(0) = u(T), u'(0) = u'(T); \]

(b) \( b = 0 \) and \( S = \mathbb{R}^N \times \mathbb{R}^N \) we get Neumann type boundary conditions \( u'(0) = u'(T) = 0; \)

(c) \( b(z) = \frac{1}{2}(A z, z)_{\mathbb{R}^{2N}}, z \in \mathbb{R}^{2N}, \) where \( A \) is a symmetric, positive \( 2N \times 2N \) real valued matrix, and \( S = \mathbb{R}^N \times \mathbb{R}^N; \) we get the following mixed boundary conditions

\[ \begin{pmatrix} h_p(u')(0) \\ -h_p(u')(T) \end{pmatrix} = A \begin{pmatrix} u(0) \\ u(T) \end{pmatrix}. \]

For these choices of \( F, b \) and \( S \) it follows by Corollary 3.1 that for each fixed \( \gamma, \mu > 0 \), there exists an open interval \( \Lambda_0 \subset [0, +\infty[ \) such that for each \( \lambda \in \Lambda_0 \) the problem \((P_{\lambda, \mu})\) has at least two nontrivial solutions.

4. Second Type Problem

Theorem 4.1. Let \( F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a function satisfying \((F_1) - (F_5)\) and let \( j : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a convex function. Then, there exist a non-degenerate compact interval \([a, b] \subset [0, +\infty[ \) and a number \( \sigma_0 > 0 \) such that for every \( \lambda \in [a, b] \) there exists \( \mu_0 > 0 \) such that for each \( \mu \in ]0, \mu_0[, \) the problem \((P_{\lambda, \mu})\) has at least three solutions with norms less than \( \sigma_0 \). Moreover, if \( 0 \notin \partial j(0, 0) \), then these solutions are nontrivial.
proof. We define the function \( g : [0, +\infty[ \rightarrow \mathbb{R}, \) by

\[
g(t) = \sup \{-\mathcal{F}(u) : \varphi_\gamma(u) \leq t \}, \text{ for all } t > 0.
\]

Using (2.3) for \( r \in ]p, p^*[ \) it follows that for all \( u \in W^{1,p} \) we have

\[
-\mathcal{F}(u) \leq \frac{\varepsilon}{\gamma} ||u||_\gamma^p + \frac{\bar{c}r}{\delta^r-1} ||u||_\gamma^r.
\]

Since \( p < r, \) this implies

\[
\lim_{t \to 0^+} \frac{g(t)}{t} = 0.
\]

As in the proof of Theorem 3.1, by \((F_5)\) there exists \( u_0 \in W^{1,p} \setminus \{0\} \) such that \(-\mathcal{F}(u_0) > 0. \) Due to the convergence relation above, it is possible to choose a real number \( t_0 \) such that \( 0 < t_0 < \varphi_\gamma(u_0) \) and

\[
\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).
\]

We choose \( \rho_0 > 0 \) such that

\[
g(t_0) < \rho_0 < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)) t_0.
\]

We apply Theorem 5.3 to the space \( W^{1,p} \), the interval \( I = ]0, +\infty[ \) and the function \( \Psi : W^{1,p} \times I \rightarrow \mathbb{R} \) defined by

\[
\Psi(u, \lambda) = \varphi_\gamma(u) + \lambda (\rho_0 + \mathcal{F}(u)), \text{ for all } (u, \lambda) \in W^{1,p} \times I
\]

and \( \Phi : W^{1,p} \rightarrow \mathbb{R} \) by

\[
\Phi(u) = J(u) \text{ for all } u \in W^{1,p}.
\]

Clearly, by Proposition 2.2 \( \Psi(\cdot, \lambda) \) and \( \Phi \) are sequentially weakly l.s.c. for all \( u \in W^{1,p}. \) Moreover, \( \Psi(\cdot, \lambda) \) is continuous (the norm \( \varphi_\gamma \) and \( \mathcal{F} \) are continuous functions), coercive (by Proposition 2.2), and obviously \( \Psi(u, \cdot) \) is concave for all \( u \in W^{1,p}. \)

By the same technique as in the proof of Theorem 3.1 we prove the minimax inequality

\[
\sup_{\lambda \in I} \inf_{u \in W^{1,p}} \Psi(u, \lambda) < \inf_{u \in W^{1,p}} \sup_{\lambda \in I} \Psi(u, \lambda).
\]

Note, that the role of the function \( \varphi_\gamma + J + \lambda \mathcal{F} + \rho_0 \lambda \) from Theorem 3.1 is now replaced by \( \Psi(\cdot, \lambda). \)

We can apply Theorem 5.3. Fix \( \delta > \eta_1, \) and for every \( \lambda \in I \) denote

\[
S_\lambda = \{ u \in W^{1,p} : \Psi(u, \lambda) < \delta \}.
\]
There exists a non-empty open set $I_0 \subset ]0, +\infty[ $ with the following property: for every $\lambda \in I_0$ there exists $\lambda_0 > 0$, such that for each $\mu \in ]0, \mu_0[$, the functional
\[
u \mapsto \Psi(\nu, \lambda) + \mu \Phi(\nu)
\]
has at least two local minima lying in the set $S_\lambda$. Let $[a, b] \subset I_0$ be a non-degenerate compact interval.

We prove now the assertion of our theorem: Let $\lambda \in [a, b]$ be a real number. From what stated above, there exists $\mu_0 > 0$ such that for all $\mu \in ]0, \mu_0[$ the functional $E(\lambda, \mu, \cdot)$ admits at least two local minima $u_{\lambda, \mu}^1$, $u_{\lambda, \mu}^2 \in S_\lambda$, therefore by Proposition 5.1 (for $G(u) = \lambda F(u), \psi(u) = \varphi_\gamma(u) + \mu J(u), u \in W^{1, p}$) these are critical points of $E(\lambda, \mu, \cdot)$.

Observe that $S := \bigcup_{\lambda \in [a, b]} S_\lambda \subseteq S_a \cup S_b$.

Since $\Psi(\cdot, \lambda)$ is coercive (see Proposition 2.2 applied for $E(\lambda, 0, \cdot)$), the latter sets are bounded, hence $S$ is bounded as well. By choosing $\sigma_0 > \sup_{u \in S} \|u\|_{\gamma_1}$, we get
\[
\|u_{\lambda, \mu}^1\|_{\gamma_1}, \|u_{\lambda, \mu}^2\|_{\gamma_1} < \sigma_0.
\]

To prove the existence of a third critical point for $E(\lambda, \mu, \cdot)$, we apply Proposition 5.2 (for $G(u) = \lambda F(u) + \varphi_\gamma(u) + \mu J(u), \psi(u) = 0, u \in W^{1, p}$; note that, since $J$ is convex and continuous, it is then also locally Lipschitz), since the (PS) condition holds by Proposition 2.2. Finally, by Proposition 2.1 it follows that these critical points are solutions of $\left(P_{\lambda, \mu}\right)$.

Obviously, if $0 \notin \partial j(0, 0)$, then each solution is nontrivial.

**Example 4.1.** We give an example of functions $F$ and $j$ that satisfy the assumptions of Theorem 4.1: Let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be defined by
\[
F(t, x) = -f(t) \cdot \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\}
\]
for all $t \in [0, T], x \in \mathbb{R}^N$,

where $\alpha > 0, \beta \in ]0, p[, f \in L^1(0, T; \mathbb{R}_+ \setminus \{0\})$, and let $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be given by
\[
j(x, y) = \max\{|(x, y) - (1, 1)|^a + 1, |(x, y) - (1, 1)|^b + 1\}
\]
for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$,

where $a > b \geq 1$ and $(1, 1) \in \mathbb{R}^N \times \mathbb{R}^N$ denotes the vector with all coordinates 1. By Theorem 4.1 it follows that in this case there exist at least three nontrivial solutions for the eigenvalue problem $(P_{\lambda, \mu})$. 

\[\blacksquare\]
5. APPENDIX - BASIC NOTIONS AND RESULTS

Let \((X, \| \cdot \|)\) be a real Banach space and \(X^*\) its topological dual. A function \(G : X \to \mathbb{R}\) is called \emph{locally Lipschitz} if each point \(u \in X\) possesses a neighborhood \(N_u\) such that \(|G(u_1) - G(u_2)| \leq L\|u_1 - u_2\|\) for all \(u_1, u_2 \in N_u\), for a constant \(L > 0\) depending on \(N_u\). The \emph{generalized directional derivative} of \(G\) at the point \(u \in X\) in the direction \(z \in X\) is

\[
G^0(u; z) = \limsup_{w \to u, s \to 0^+} \frac{G(w + sz) - G(w)}{s}.
\]

The \emph{generalized gradient} (in the sense of Clarke [1]) of \(G\) at \(u \in X\) is defined by

\[
\partial^G u = \{x^* \in X^* : \langle x^*, x \rangle \leq G^0(u; x), \forall x \in X\},
\]

where \(\langle \cdot, \cdot \rangle\) is the duality pairing between \(X^*\) and \(X\).

Let \(G : X \to \mathbb{R}\) be a locally Lipschitz function, and let \(\psi : X \to [\infty, +\infty)\) be a convex, proper, l.s.c. function.

**Definition 5.1.** [14]. An element \(u \in X\) is said to be a critical point of \(E = G + \psi\), if

\[
G^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \forall v \in X.
\]

In this case, \(E(u)\) is a critical value of \(E\).

In the case of differentiable functions one gets the notion of critical point introduced by A. Szulkin [18].

**Definition 5.2.** [14]. The functional \(E = G + \psi\) is said to satisfy the Palais-Smale condition at level \(c \in \mathbb{R}\) (shortly, \((PS)_c\)) if every sequence \(\{u_n\}\) in \(X\) satisfying \(E(u_n) \to c\) and

\[
G^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\|, \forall v \in X,
\]

for a sequence \(\{\varepsilon_n\} \subset [0, \infty)\) with \(\varepsilon_n \to 0\), contains a convergent subsequence. If \((PS)_c\) is verified for all \(c \in \mathbb{R}\), \(E\) is said to satisfy the Palais-Smale condition (shortly, \((PS)\)).

**Proposition 5.1.** [12, Proposition 2.1]. Each local minimum of \(E = G + \psi\) is necessarily a critical point of \(E\).

**Theorem 5.2.** [12, Theorem 3.1]. Assume that \(X\) is a separable and reflexive Banach space, \(\Lambda\) is a real interval, \(G, H : X \to \mathbb{R}\) are locally Lipschitz functions and \(\psi : X \to [\infty, +\infty]\) is a convex, proper, l.s.c. function, such that:
(a) for every \( \lambda \in \Lambda \) the function \( G + \psi + \lambda H \) fulfils the \((PS)\) condition, together with
\[
\lim_{\|u\| \to +\infty} \left( G(u) + \psi(u) + \lambda H(u) \right) = +\infty;
\]

(b) there exists a continuous concave function \( h : \Lambda \to \mathbb{R} \) satisfying
\[
\sup_{\lambda \in \Lambda} \inf_{u \in X} \left( G(u) + \psi(u) + \lambda H(u) + h(\lambda) \right) < \inf_{u \in X} \sup_{\lambda \in \Lambda} \left( G(u) + \psi(u) + \lambda H(u) + h(\lambda) \right).
\]

Then, there is an open interval \( \Lambda_0 \subseteq \Lambda \) such that for each \( \lambda \in \Lambda_0 \) the function \( G + \psi + \lambda H \) has at least three critical points in \( X \).

The following result is proved by Marano and Motreanu and it generalizes results of P. Pucci, J. Serrin [16]:

**Proposition 5.2.** [12, Corollary 2.1]. Let \( I = G + \psi \) satisfying the Palais-Smale condition \((PS)\). If \( E \) has two local minima \( u_0, u_1 \in X \), then it admits at least three critical points.

The main tool in our investigations is the result of B. Ricceri [17, Theorem 4], which we state for the reader’s convenience in a slightly modified form (adapted for the weak topology), suitable for our purposes:

**Theorem 5.3.** Let \( X \) be a real, reflexive, separable Banach space, let \( I \subseteq \mathbb{R} \) be an interval, and let \( \Psi : X \times I \to ]-\infty, +\infty[ \) be a function satisfying the following conditions:

1. \( \Psi(x, \cdot) \) is concave in \( I \) for all \( x \in X \);
2. \( \Psi(\cdot, \nu) \) is upper semicontinuous, coercive and sequentially weakly lower semicontinuous in \( X \) for all \( \nu \in I \);
3. \( \eta_1 := \sup_{\nu \in I} \inf_{x \in X} \Psi(x, \nu) < \inf_{x \in X} \sup_{\nu \in I} \Psi(x, \nu) =: \eta_2. \)

Then, for each \( \delta > \eta_1 \) there exists a non-empty open set \( I_0 \subseteq I \) with the following property: for every \( \nu \in I_0 \) and every sequentially weakly l.s.c. function \( \Phi : X \to \mathbb{R} \), there exists \( \tau_0 > 0 \) such that, for each \( \tau \in ]0, \tau_0[ \), the function \( \Psi(\cdot, \nu) + \tau \Phi(\cdot) \) has at least two local minima lying in the set \( \{ x \in X : \Psi(x, \nu) < \delta \} \).

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Hannelore Lisei and Csaba Varga  
Babeș-Bolyai University,  
Faculty of Mathematics and Computer Science,  
Str. Kogalniceanu nr. 1,  
RO-400084 Cluj-Napoca, Romania  
E-mail: hanne@math.ubbcluj.ro  
csvarga@cs.ubbcluj.ro

Gheorghe Moroșanu  
Department of Mathematics and Its Applications,  
Central European University,  
Nador u. 9, H-1051 Budapest,  
Hungary  
E-mail: morosanug@ceu.hu