A CLASS OF NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS*

Veli-Matti Hokkanen\(^1\) and Gheorghe Moroşanu\(^2\)
\(^1\)University of Jyväskylä,
Department of Mathematics, FIN-40351 Jyväskylä, Finland
\(^2\)"Al. I. Cuza" University,
Faculty of Mathematics, 6600 Iaşi, Romania

Abstract. A class of nonlinear parabolic boundary value problems (1.1) is studied by means of the theory of abstract Cauchy problems containing time dependent maximal monotone and compact operators whose domains may depend on time.

Key words. Nonlinear parabolic boundary value problem, maximal monotone operator, subdifferential, abstract Cauchy problem.

AMS Subject Classifications: 35K55, 34G20, 47H15.

1. Introduction

We denote \(u_x = \partial u / \partial x\), \(u_t = \partial u / \partial t\) and study the problem

\[
\begin{align*}
  u_t(t, x) - w_x(t, x) + \tilde{w}(t, x) &= f(t, x), \text{ for a.e. } x \in ]0, 1[, t \in ]0, T[, \quad (1.1a) \\
  w(t, x) &\in G(t, x)u_x(t, x), \text{ for a.e. } x \in ]0, 1[, t \in ]0, T[, \quad (1.1b) \\
  \tilde{w}(t, x) &\in K(t, x)u(t, x), \text{ for a.e. } x \in ]0, 1[, t \in ]0, T[, \quad (1.1c) \\
  (w(t, 0), -w(t, 1)) &\in \beta(t)(u(t, 0), u(t, 1)), \text{ for a.e. } t \in ]0, T[, \quad (1.1d) \\
  u(0, x) &= u_0(x), \text{ for a.e. } x \in ]0, 1[, \quad (1.1e)
\end{align*}
\]

where \(T > 0\) and \(G(t, x), K(t, x) \subset \mathbb{R} \times \mathbb{R}\) and \(\beta(t) \subset \mathbb{R}^2 \times \mathbb{R}^2\) are maximal monotone operators. This very general model describes heat conduction and diffusion phenomena. For \(G\) independent on \(t\) and linear in \(u_x\), see [M1], [MP] (see also [M2] for time dependent boundary conditions). Equations with nonlinear terms in \(u_x\) are investigated in [GL], [L], [LF] (see also their references). Equations like (1.1) in more dimensions are considered in [H], [T], ...

Here we study (1.1) by using abstract Cauchy problems (1.2) in a real Hilbert space with maximal monotone operators \(A(t)\), perturbed by operators \(B(t)\):

\[
u'(t) + A(t)u(t) + B(t)u(t) \ni f(t), \quad t > 0, \quad u(0) = u_0. \quad (1.2)
\]

In Section 3 we consider (1.1) as (1.2) with time independent \(A(t)\) and \(B(t) \equiv 0\). In Section 4 we study the existence of a solution for (1.2). Our condition (H.3) allowing

*This research is supported by The Academy of Finland and by The Romanian Academy.
A(t) to be a subdifferential with time dependent domain, is close to that of [AB]; our proofs are based on the methods used for the case of time independent A(t) in [Br]; the perturbed problems are handled by fixed point theorems. In Section 5 our results will be applied to (1.1) with unbounded β(t).

2. The notation

Let H be a real Hilbert space, ‖·‖_H its norm and (·,·)_H its inner product. An operator A ⊂ H × H is monotone if (y_2 − y_1, x_1 − x_2)_H ≥ 0, for each (x_1, y_1), (x_2, y_2) ∈ A. It is maximal monotone if it is not contained by any other monotone operator of H. Let λ > 0, ψ : H → [−∞, ∞] be convex and A be maximal monotone in H. The subdi erential of ψ is denoted by ∂ψ and ψ_λ is the Yosida Moreau regularization of ψ. The resolvent and the Yosida approximate of A are denoted by J_λ and by A_λ, respectively; A_0x is the element of Ax with the minimal norm. Indeed,

\[ \partial \psi = \{(u, v) \in H \times H \mid \psi(u) < \infty, \psi(u) + (v, \xi - u)_H \leq \psi(\xi) \text{ for each } \xi \in H \}, \]

\[ J_\lambda = (I + \lambda A)^{-1}, A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \psi_\lambda(x) = \inf\{\frac{1}{2\lambda}\|y - x\|_H^2 + \psi(y) \mid y \in H\}. \]

The positive part of ψ is denoted by ψ_+. For the further details and the theory of differential equations containing maximal monotone operators, we refer to [Br], [Ba], [BP] and [M1]. For the general functional analysis, the reader may see [KA].

3. The case of time independent G and K

In this section we study problem (3.1) with non-homogenous boundary conditions;

\[ \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} G(x, u_x) + K(x, u) = f(t, x), \quad 0 < x < 1, \quad t > 0, \tag{3.1a} \]

\[ s(t) + (G(0, u_x(t, 0)), -G(1, u_x(t, 1))) \in \beta(u(0), u(t, 1)), \quad t > 0, \tag{3.1b} \]

\[ u(0, x) = u_0(x), \quad 0 < x < 1. \tag{3.1c} \]

Let us introduce our main assumptions. Let δ > 0.

(I_1) \ G \in C^1([0, 1] \times \mathbb{R}) and \ G_x(x, \xi) ≥ \delta, for each \ x \in [0, 1], \ \xi \in \mathbb{R}.

(I_2) \ K \in C([0, 1] \times \mathbb{R}) and \ K(x, ·) \text{ is monotone, for each } x \in [0, 1].

(I_3) \ β \subset \mathbb{R}^2 \times \mathbb{R}^2 \text{ is a maximal monotone operator.}

(I_4) \ j : \mathbb{R}^2 \mapsto [−\infty, \infty] \text{ is proper, convex, and lower semicontinuous; } \beta = \partial j.

It is well known [M1] that (3.1b) includes many classical boundary value conditions: Dirichlet, Neumann, Robin-Steklov, periodic, etc.

Define H = L^2(0, 1), u(t) = u(t, ·), f(t) = f(t, ·) and A(t) : D(A(t)) → H;

\[ D(A(t)) = \{v \in H^2(0, 1) \mid s(t) + (G(0, v'(0)), -G(1, v'(1))) \in \beta(v(0), v(1))\}, \tag{3.2a} \]

\[ A(t)v(x) = -\frac{d}{dx} G(x, v'(x)) + K(x, v(x)), \quad 0 < x < 1, \quad t ≥ 0. \tag{3.2b} \]
If \( s \equiv 0 \), we denote \( A = A(t) \). Now, (3.1) is a Cauchy problem (3.3);
\[
  u'(t) + A(t)u(t) \ni f(t), \ t > 0, \ u(0) = u_0.
\]  

**Proposition 3.1.** Assume \( I_1, I_2 \) and \( I_4 \). Let \( s \equiv 0 \). Then \( \phi : H \mapsto [-\infty, \infty] \),
\[
  \phi(v) = \begin{cases} \int_0^1 (g(\cdot, v') + k(\cdot, v)) \, dx + j(v(0), v(1)), \text{ if } v \in H^1(0,1), \\ \infty, \text{ otherwise}, \end{cases} \tag{3.4a}
\]
\[
  g(x, \xi) = \int_0^\xi G(x, \tau) \, d\tau, \quad k(x, \xi) = \int_0^\xi K(x, \tau) \, d\tau, \tag{3.4b}
\]
is proper, convex, and lower semicontinuous, \( A = \partial\phi \), and \( A \) is densely defined.

**Proof.** By \( I_1 \), \( G(0, \cdot) \) and \( G(1, \cdot) \) are surjective. So, there exist \( a, b, c, d \in \mathbb{R} \) such that \( (G(0, c), -G(1, d)) \in \beta(a, b) \). Therefore \( \emptyset \neq D(A) \), since it contains \( \hat{v} \),
\[
  \hat{v}(x) = (2a - 2b + c + d)x^3 + (-3a + 3b - 2c - d)x^2 + cx + a. \tag{3.5}
\]
Hence \( \{ \hat{v} + \psi \mid \psi \in C_0^\infty([0,1]) \} \subset D(A) \). Thus \( D(A) \) is dense in \( H \).

By Lemma 5.1 below, \( \phi \) is proper, convex and lower semicontinuous.

Clearly, \( A \subset \partial\phi \). Let us show that \( \partial\phi \subset A \). So, let \( (u, v) \in \partial\phi \). Then by Lemma 5.1 below, \( u \in H^1(0,1) \) and there is \( w \in H^1(0,1) \) such that
\[
  v(x) = -w'(x) + K(x, u(x)) = (Aw)(x), \text{ for a.e. } x \in ]0,1[, \tag{3.6a}
\]
\[
  w(x) = G(x, u'(x)), \text{ for a.e. } x \in ]0,1[, \tag{3.6b}
\]
\[
  (w(0), -w(1)) \in \beta(u(0), u(1)). \tag{3.6c}
\]
By \( I_1 \) and by the Implicit Function Theorem, \( x \mapsto G(x, \cdot)^{-1}\xi \) belongs to \( C^1([0,1]) \), for each \( \xi \in \mathbb{R} \). Since \( G(x, \cdot)^{-1} \) are \( \frac{1}{d} \)-Lipschitzian and \( w \in H^1(0,1) \), then \( x \mapsto w'(x) = G(x, \cdot)^{-1}w(x) \) belongs to \( H^1(0,1) \). Hence \( u \in H^2(0,1) \). Proposition 3.1 is proved.

**Proposition 3.2.** Assume \( I_1, I_2 \) and \( I_3 \). Then \( A(t) \) given by (3.2) is a maximal monotone operator.

**Proof.** By Remark 5.2 below, the operator given by (5.2) is maximal monotone; we denote it by \( \hat{A}(t) \). Clearly, \( \hat{A}(t) \subset A(t) \). Let \( v \in H \). Then there are \( u, w \in H^1(0,1) \) satisfying (3.6). Thus \( u \in H^2(0,1) \) and \( A(t) = \hat{A}(t) \).

**Remark 3.1.** Propositions 3.1 and 3.2 improve the results of [GL] and of [AM].

**Theorem 3.1.** (Existence and uniqueness if \( \beta = \partial j \)). Assume \( I_1, I_2 \) and \( I_4 \), let \( T > 0, u_0 \in H, f \in L^2(Q_T), Q_T = ]0, T[ \times ]0, 1[ \) and \( s \in \mathbb{R}^2 \). Then (3.1) has a unique strong solution \( u \in C([0,T]; H) \) with \( (t, x) \mapsto \sqrt{t}u_t(t, x) \) in \( L^2(Q_T) \). If, in addition, \( u_0 \in D(\phi) \), then \( u \in H^1(Q_T) \). If, in addition, \( u_0 \in D(A) \) and \( f \in W^{1,1}(0, T; H) \), then \( u \in L^\infty(0, T; H^1(0,1)) \cap L^2(Q_T) \).
Proof. The first part is implied by [Br, Thm. 3.6] (see also Proposition 3.1 above). If \( u_0 \in D(\phi) \), we have from the same theorem that \( u_t \in L^2(Q_T) \), so \( Au = f - u_t \in L^2(Q_T) \). Let \( \hat{v} \) be given by (3.5). By I₂, for a.e. \( t \in ]0, T[ \),

\[
\delta \int_0^1 (u_x(t, x) - \hat{v}'(x))^2 \, dx \leq \int_0^1 \left( f(t, x) - u_t(t, x) - A\hat{v}(x) \right) (u(t, x) - \hat{v}(x)) \, dx. \tag{3.6}
\]

Hence \( u \in H^1(Q_T) \). Finally, if \( u_0 \in D(A) \) and \( f \in W^{1,1}(0, T; L^2(0, 1)) \), then \( u_t \in L^\infty(0, T; H) \) (see [M1, p. 48]). So, using (3.6), we can see that \( u_x \in L^\infty(0, T; H) \) and hence \( u \in L^\infty(0, T; H^1(0, 1)) \). Then \( u \in L^\infty(Q_T) \), by

\[
u(t, x) = \int_0^1 \left( \tau \frac{\partial u}{\partial \tau}(t, \tau) + u(t, \tau) \right) d\tau - \int_x^1 \frac{\partial u}{\partial \tau}(t, \tau) d\tau. \tag{3.7}
\]

Theorem 3.1 is proved.

Remark 3.2. If I₁, I₂ and I₄ are fulfilled, \( u_0 \in H \) and \( f \in L^1(0, T; H) \), then problem (3.1) has a unique weak solution.

Proposition 3.3. If I₁, I₂ and I₃ are satisfied and if \( s(t) \equiv 0 \), then the resolvent of \( A \), \( J_\lambda = (I + \lambda A)^{-1} : H \mapsto H \), is compact, for every \( \lambda > 0 \).

Proof. Let \( \lambda > 0 \) and \( Y \subset H \) be bounded. Denote \( u_p = (I + \lambda A)^{-1} p \), for each \( p \in Y \). Let \( \hat{v} \) be given by (3.5). Then there is \( M' > 0 \), independent on \( p \);

\[
M' + \frac{1}{2} \| u_p - \hat{v} \|_H^2 \geq (p - \hat{v} - \lambda A\hat{v}, u_p - \hat{v})_H \geq \| u_p - \hat{v} \|_H^2 + \lambda \delta \| u_p' - \hat{v}' \|_H^2.
\]

Hence the resolvent is bounded as \( H \mapsto H^1(0, 1) \), so it is compact as \( H \mapsto H \).

Theorem 3.2. (Asymptotic behaviour if \( \beta = \partial j \)). Assume I₁, I₂ and I₃. Let \( u_0 \in H \), \( f \in L^1(0, \infty; H) \), \( s(t) \equiv 0 \), \( F := A^{-1} \neq \emptyset \), and let \( u \) be the weak solution of (3.1). Then there exists \( \hat{p} \in F \) such that \( u(t) \to \hat{p} \) in \( H \), as \( t \to \infty \). If, in addition, \( f \in W^{1,1}(0, \infty; H) \), then \( u(t) \to \hat{p} \) weakly in \( H^1(0, 1) \) and strongly in \( C([0, 1]) \), as \( t \to \infty \).

Proof. For \( f \in W^{1,1}(0, \infty; H) \) it follows by Proposition 3.3 that the trajectory \( \{ u(t) | t \geq \epsilon \} \) is bounded in \( H^1(0, 1) \), for each \( \epsilon > 0 \). Indeed,

\[
u(t) = (I + A)^{-1} \left( f(t) + u(t) - \frac{d^+ u}{dt}(t) \right) \tag{3.8}
\]

and the set \( \{ u(t) | t \geq 0 \} \) is bounded in \( H \), because \( F \neq \emptyset \) (see [M1, p. 73]). For the rest of the proof see [M1, Chapters II, III].

Remark 3.3. For \( \beta \) not subdifferential, but \( s(t) \equiv 0 \), and \( G(x, \cdot) \) linear, see [M1] and [MP]. Still \( A \) is maximal monotone in \( H \).

Next, we study nonhomogeneous boundary conditions. As both \( G \) and \( \beta \) are non-linear, (3.1c) cannot be homogenized. So (3.1) has time dependent \( D(A(t)) \).
We recall that \( u \in C([0,T];H) \) is said to be a weak solution of (3.3) on \([0,T]\) (cf. [Br, p. 64]), if there exist \( u_{n0} \in H, f_n \in L^1(0,T;H) \), maximal monotone \( A_n(t) \subset H \times H \) and \( u_n \in W^{1,\infty}(0,T;H) \), \( n = 1, 2, \ldots \), satisfying

\[
\begin{align*}
    u_n' + A_n(t)u_n & \ni f_n(t), \quad \text{for a.e.}\ t \in [0,T],\ u_n(0) = u_{n0}, \quad (3.9) \\
u_n & \rightarrow u \text{ in } C([0,T];H) \text{ and } f_n \rightarrow f \text{ in } L^1(0,T;H), \text{ as } n \rightarrow \infty. \quad (3.10)
\end{align*}
\]

**Theorem 3.3.** (Existence of weak solutions). Assume I₁, I₂, and I₃. Let \( u_0 \in H, f \in L^1(0,T;H), \) and \( s \in L^2(0,T;\mathbb{R}^2) \). Then (3.3) has a unique weak solution.

**Proof.** Let \( u_{n0} \in D(A), f_n \in W^{1,\infty}(0,T;H), s_n \in W^{1,\infty}_0(0,T;\mathbb{R}^2) \), \( \tilde{A}_n(t) = A_n(t) - f_n(t) \), where \( A_n(t) \) is \( A(t) \) given by (3.2) with \( s_n(t) \) instead of \( s(t) \). We assume that

\[
u_{n0} \rightarrow u_0 \quad \text{in } H, \quad s_n \rightarrow s \quad \text{in } L^2(0,T;\mathbb{R}^2), \quad f_n \rightarrow f \quad \text{in } L^1(0,T;H), \quad \text{as } n \rightarrow \infty. \quad (3.11)
\]

By Prop. 3.2 each \( \tilde{A}_n(t) \) and \( A_n(t) \) is maximal monotone. We recall that

\[

\|(\xi(0),\xi(1))\|_{\mathbb{R}^2} \leq 2\|\xi'\|_H + 2\|\xi\|_H, \quad \text{for each } \xi \in H^1(0,1). \quad (3.12)
\]

Let \( \sigma, \tau \in [0,T], \) \( v \in D(\tilde{A}_n(\tau)) \), and \( w \in D(\tilde{A}_n(\sigma)) \). By (3.2), I₁ and by (3.12),

\[
\begin{align*}
- (v - w, \tilde{A}_n(\tau)v - \tilde{A}_n(\sigma)w) & \leq - (G(x, v'(x)) - G(x, w'(x))(v(x) - w(x)))  \\
 & \leq - \delta \| v' - w' \|_H^2 + \cdots \leq 2\|v - w\|_H^2 + (\tau - \sigma)^2 \left( \frac{1 + \delta}{\delta} \left\| s_n' \right\|_{L^\infty(0,T;\mathbb{R}^2)}^2 + \left\| f_n' \right\|_{L^\infty(0,T;H)}^2 \right).
\end{align*}
\]

Hence we obtain from [T, pp. 147, 138-9, 142-3] that the problems (3.9) have the solution \( u_n \in W^{1,\infty}(0,T;H) \). By (3.9), (3.2), I₁ and by (3.12), for a.e. \( t \in [0,T] \),

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| u_n(t) - u_m(t) \|_H^2 + \frac{\delta}{2} \| u_{n,xx}(t) - u_{m,xx}(t) \|_H^2 & \leq \frac{2}{\delta} \left\| s_n(t) - s_m(t) \right\|_{\mathbb{R}^2}^2 + \left\| u_n(t) - u_m(t) \right\|_H \left( 2\left\| s_n(t) - s_m(t) \right\|_{\mathbb{R}^2} + \left\| f_n(t) - f_m(t) \right\|_H \right).
\end{align*}
\]

We integrate and apply a Gronwall type inequality [Br, p. 157]. Then

\[
\| u_n(t) - u_m(t) \|_H \leq \| u_{n0} - u_{m0} \|_H + \frac{2}{\sqrt{\delta}} \| s_n - s_m \|_{L^2(0,T;\mathbb{R}^2)} + \left( 2\| s_n - s_m \|_{L^1(0,T;\mathbb{R}^2)} + \| f_n - f_m \|_{L^1(0,T;H)} \right).
\]

for each \( t \in [0,T], \) \( m, n = 1, 2, \ldots \) This completes the proof.

**Remark 3.4.** By (3.13), the weak solution \( u \in C([0,T];H) \cap L^2(0,T;H^1(0,1)) \).
Theorem 3.4. (Asymptotic behaviour). Assume $I_1$, $I_2$, $I_3$, and let $u_0 \in H$, $f \in L^1(0, \infty; H)$, $s \in L^2(0, \infty; \mathbb{R}^2) \cap L^1(0, \infty; \mathbb{R}^2)$, $u \in L^2_{loc}([0, \infty; H^1(0, 1))] \cap C([0, \infty; H)$, where $u$ is the weak solution of (3.3). Let all the weak solutions of (3.3) with $f \equiv 0$, $s \equiv 0$ converge in $H$, as $t \to \infty$. Then $u(t) \to p \in A^{-1}0$ in $H$, as $t \to \infty$.

Proof. Let $u_n$ be the weak solution of (3.3) with $(f_n, s_n)$ instead of $(f, s)$,

$$f_n(t) = \begin{cases} f(t), & \text{if } 0 < t < n, \\ 0, & \text{if } t \geq n, \end{cases} \quad s_n(t) = \begin{cases} s(t), & \text{if } 0 < t < n, \\ 0, & \text{if } t \geq n, \end{cases} \quad n = 1, 2, \ldots$$

We have (3.14) for the approximates of $(u_n, f_n, u_0)$ and of $(u, f, u_0)$. Taking the limit we obtain, for each $t > 0$ and $n = 1, 2, \ldots$,

$$\|u_n(t) - u(t)\|_H \leq \frac{2}{\sqrt{\delta}}\|s\|_{L^2(n, \infty; \mathbb{R}^2)} + 2\|s\|_{L^1(n, \infty; \mathbb{R}^2)} + \|f\|_{L^1(n, \infty; H)}.$$ (3.15)

There is $p_n \in H$ such that $u_n(t) \to p_n$ in $H$, as $t \to \infty$. By (3.14), $(p_n)$ is a Cauchy sequence, converging in $H$ toward some $p \in H$. Then by (3.15),

$$\|u(t) - p\|_H \leq \|u_n(t) - p_n\|_H + \|u(t) - u_n(t)\|_H + \|p_n - p\|_H < \epsilon + \|u_n(t) - p_n\|_H,$$

for any $\epsilon > 0$ if $n$ is large enough. Therefore,

$$\limsup_{t \to \infty} \|u(t) - p\|_H \leq \epsilon, \quad \text{for each } \epsilon > 0.$$

Hence $u(t) \to p$ in $H$, as $t \to \infty$. Theorem 3.4 is proved.

4. Non-autonomous Cauchy problem

Let $H$ be a real Hilbert space, $M, T > 0$ and $\eta \in L^1(0, T)$. For $\phi : [0, T] \times H \to [-\infty, \infty]$ and $B(t) \subset H \times H, t \in [0, T]$, we formulate:

(H.1) For each $t \in [0, T]$, $\phi(t, \cdot)$ is a proper convex lower semicontinuous function and $B(t)$ is a maximal monotone operator in $H$.

(H.2) There is $z \in L^2(0, T; H)$ such that $z(t) \in D(\partial\phi(t, \cdot))$ for a.e. $t \in [0, T]$, and $t \mapsto \phi(t, z(t)), t \mapsto \|\partial\phi^0(t, \cdot)z(t)\|_H^2$ and $t \mapsto \|\partial^0(t, z(t)\|_H^2$ are integrable.

(H.3) For each $\lambda \in [0, 1]$ and $y \in H$, $\phi_\lambda(\cdot, y) \in W^{1,1}(0, T)$ and a.e on $[0, T]$, $\frac{\partial\phi_\lambda}{\partial t}(t, y) \leq \frac{1}{3}\|\partial\phi_\lambda(t, \cdot)y\|_H^2 + M\|y\|_H^2 + \eta(t)(1 + \phi_\lambda(t, y)) \geq \|B_\lambda(t)y\|_H^2$.

(H.4) For each $y \in H$ and $\lambda \in [0, 1]$, the mappings $t \mapsto (I + \lambda\partial\phi(t, \cdot))^{-1}y$ and $t \mapsto (I + \lambda B(t))^{-1}y$ are measurable.
Theorem 4.1. Assume (H.1)-(H.4). If \( u_0 \in D(\phi(0, \cdot)) \) and \( f \in L^2(0, T; H) \), then there exist \( v, w \in L^2(0, T, H) \), a unique \( u \in H^1(0, T; H) \), and some constants \( M_1, M_2 > 0 \), independent on \( f \) and \( u_0 \), such that

\[
\begin{align*}
&u'(t) + v(t) + w(t) = f(t), \text{ for a.e. } t \in [0, T], \quad (4.1a) \\
v(t) \in \partial \phi(t, \cdot)u(t), \quad (4.1b) \\
w(t) \in B(t)u(t), \text{ for a.e. } t \in [0, T], \\
u(0) = u_0;
\end{align*}
\]

\[
\|u\|_{L^\infty(0, T; H)}^2 + \int_0^T \left( \|u'(\tau)\|_H^2 + \|v(\tau)\|_H^2 + \|w(\tau)\|_H^2 \right) d\tau + \text{ess sup}_{\tau \in [0, T]} \phi_+(\tau, u(\tau))
\leq M_1 \int_0^T \|f(\tau)\|_H^2 d\tau + M_2 (1 + \|u_0\|_H^2 + \phi_+(0, u_0)). \tag{4.2}
\]

Theorem 4.2. Assume (H.1)-(H.4). Let \( u_0 \in \overline{D(\phi(0, \cdot))} \), \( f \in L^1(0, T; H) \) and

\[
M_f = \int_0^T \tau \|f(\tau)\|_H^2 d\tau + \|f\|_{L^1(0, T; H)}^2, \quad M_{u_0} = \int_0^T \frac{1}{\tau} \|z(\tau) - u_0\|_H^2 d\tau + \|u_0\|_H^2,
\]

be finite. Then there exist measurable \( v, w : [0, T] \mapsto H \), a unique \( u \in C([0, T]; H) \), differentiable a.e. on \([0, T]\), and constants \( M_3, M_4 > 0 \), independent on \( u_0 \) and \( f \). They satisfy (4.1) and (4.3);

\[
\begin{align*}
&\|u\|_{L^\infty(0, T; H)}^2 + \int_0^T \tau \left( \|u'(\tau)\|_H^2 + \|v(\tau)\|_H^2 + \|w(\tau)\|_H^2 \right) d\tau + \text{ess sup}_{\tau \in [0, T]} \tau \phi_+(\tau, u(\tau)) \\
&\leq M_3 M_f + M_4 M_{u_0} + M_4. \tag{4.3}
\end{align*}
\]

We state for \( C(t) : D(C(t)) \mapsto H, D(C(t)) \subset H, t \in [0, T] \), \( \alpha \geq 0, \beta > 0 \): \( (H.5) \) For each \( y \in C([0, T]; H) \) with \( y(t) \in D(C(t)), t \in [0, T] \), the mappings \( t \mapsto C(t)y(t) \) and \( t \mapsto t^\alpha \|C(t)y(t)\|_H^2 \) are integrable.

\( (H.6) \) For a.e. \( t \in [0, T] \), \( D(C(t)) = \overline{D(\phi(t, \cdot))} \), and

\[
\|C(t)x - C(t)y\|_H \leq \eta(t)\|x - y\|_H, \text{ for each } x, y \in D(C(t)).
\]

\( (H.7) \) The closure of \( \{y \in H \mid \|y\|_H^2 + \phi(t, y) \leq \delta \} \), for a.e. \( t \in [0, T] \), is compact in \( H \), for each \( \delta > 0 \).

\( (H.8) \) There is \( \psi : [0, T] \times H \mapsto [0, \infty] \), measurable with respect to the \( \sigma \)-field generated by the products of Lebesgue sets in \([0, T]\) and of Borel sets in \( H \). For a.e. \( t \in [0, T] \) and each \( y \in D(\phi(t, \cdot)), \psi(t, \cdot) \) is proper, convex and lower semicontinuous, \( D(\psi(t, \cdot)) \subset D(C(t)) \), and

\[
\beta\|C(t)y\|_H^2 - T^{-1-\alpha}\|y\|_H^2 - \eta(t) \leq \psi(t, y) \leq \|\partial \phi^0(t, \cdot)y\|_H^2 + \|B^0(t)y\|_H^2 + \eta(t).
\]
(H.6) \( y \mapsto Cy \), \((Cy)(t) = C(t)y(t)\), is strongly-weakly closed in \( L^2(0, T; H) \).

(H.10) For a.e. \( t \in ]0, T[ \) and for each \( y \in D(\partial \phi(t, \cdot)) \),

\[
\|C(t)y\|^2_H \leq \frac{\alpha(t,y)}{\beta \|y\|_{L^1(0,T)}} + \frac{1}{\beta} \left( \frac{\|y\|_{L^1(0,T)}}{\|y\|_{L^1(0,T)}} \right)^2 + \eta(t)^2.
\]

**Theorem 4.3.** Assume (H.5) with \( \alpha = 0 \), the conditions of Theorem 4.1, and, in addition, (H.6) or (H.7)-(H.10) with \( \beta = 3M_1 \). Then there exist \( u \in H^1(0, T; H) \) and \( v, w \in L^2(0, T; H) \) which satisfy (4.4);

\[
\begin{align*}
&u'(t) + v(t) + w(t) + C(t)u(t) = f(t), \text{ for a.e. } t \in ]0, T[, \quad (4.4a) \\
v(t) \in \partial \phi(t, \cdot)u(t), w(t) \in B(t)u(t), \text{ for a.e. } t \in ]0, T[, \quad (4.4b) \\
u(0) = u_0. \quad (4.4c)
\end{align*}
\]

If, in addition, (H.6) is satisfied, then \( u \) is unique.

**Theorem 4.4.** Assume (H.5) with \( \alpha = 1 \), the conditions of Theorem 4.2, and, in addition, (H.6) or (H.7)-(H.10) with \( \beta = 4M_3 \). Then there exist \( u \in C([0, T]; H) \), differentiable a.e. on \([0, T]\), and measurable \( v, w : [0, T] \mapsto H \) which satisfy (4.4) and

\[
\int_0^T \left( \|u'(t)\|^2_H + \|v(t)\|^2_H + \|w(t)\|^2_H \right) \, dt < \infty.
\]

If, in addition, (H.6) is satisfied, then \( u \) is unique.

**Remark 4.1.** Condition (H.3) allows the domain of \( \partial \phi(t, \cdot) \) to depend on time. Indeed, let \( H = \mathbb{R} \), \( c_1, c_2 \in H^1(0, 1) \) with \( c_1 \leq c_2 \) and \( \phi : [0, 1] \times \mathbb{R} \mapsto \{0, \infty\} \),

\[
\phi(t, x) = \begin{cases} 0, & \text{if } c_1(t) \leq x \leq c_2(t), \\ \infty, & \text{otherwise}. \end{cases}
\]

Then, for each \( x \in \mathbb{R} \) and for a.e. \( t > 0 \), \( \phi_\lambda(\cdot, x) \in W^{1,1}(0, 1) \) and

\[
\frac{\partial \phi_\lambda}{\partial t}(t,x) \leq \frac{1}{3} |\partial \phi_\lambda(t, \cdot)x|^2 + \frac{3}{4} \max(c'_1(t)^2, c'_2(t)^2).
\]

**Remark 4.2.** The estimates for \( (u, v, w) \) in Theorems 4.3 and 4.4 can be obtained from (4.2) and (4.3) by substituting \( f \) by \( f - Cu \).

**Remark 4.3.** The coefficient \( 1/3 \) in (H.3) can be slightly increased, depending on \( B(t) \). If \( B(t) \equiv 0 \), then \( 1/3 \) can be replaced by any \( \alpha \in ]0, 1[ \).

**Proof of Theorem 4.1.** We modify a textbook proof for the case of autonomous equation, see [M1, pp. 46-61], cf. [Br, pp. 54-57, 72-78]. We denote constants, independent on \( f, \lambda, t \) and \( u_0 \), by \( M_5, M_6, \ldots \).
Lemma 4.1. Let $\phi(t, \cdot) = \|\nabla \phi(t, \cdot)\|_{\infty}^2$ be Lipschitzian and $t \mapsto \partial \phi(t, \cdot) u(t) + B(\tau(t), \cdot)$ be square integrable for each $t \in [0, T]$. Then, for a.e. $t \in [0, T]$, we have
\[
u^t(t) + \partial \phi(t, \cdot) u(t) + B(\tau(t), \cdot) = f(t),
\]
where $\nu^t(0) = u_0$. 

Proof. The monotonicity of $B$ and of $\partial \phi$ is clear. The maximality is implied by $R(I + B) = R(I + \partial \phi) = L^2(\delta, T; H)$, which holds since $t \mapsto (I + \partial \phi(t, \cdot))^{-1}y(t)$ and $t \mapsto (I + B(t))^{-1}y(t)$ are square integrable, for each $y \in L^2(\delta, T; H)$.

Lemma 4.2. There are $M_1, M_2 > 0$, independent on $\lambda$, $f$ and on $u_0$, such that
\[
\|u_\lambda\|_{L^\infty(0, T; H)} + \int_0^T \left(\|u'_\lambda(t)\|_H^2 + \|\partial \phi(\lambda(t)) u_\lambda(t)\|_H^2 + \|B(\tau(t), \cdot) u_\lambda(t)\|_H^2\right) dt + \sup_{\tau \in [0, T]} \phi_\lambda(\tau, u_\lambda(t)) \leq M_1 \int_0^T \|f(t)\|_H^2 + M_2 (1 + \|u_0\|_H^2 + \phi_+(0, u_0)).
\]

Proof. By the chain rule, we have
\[
\frac{d}{dt} \phi_\lambda(t, u_\lambda(t)) = (\partial \phi_\lambda(t, \cdot) u_\lambda(t), u'_\lambda(t)) + \frac{\partial \phi_\lambda}{\partial t}(t, u_\lambda(t)).
\]
We multiply (4.5a) by $u'_\lambda(t)$, by $100 \partial \phi_\lambda(t, \cdot) u_\lambda(t)$, and by $B(\tau(t), u_\lambda(t))$, successively. By summing the results, by (4.6) and (H.3),
\[
\int_0^t \left(\frac{1}{303} \|u'_\lambda(\tau)\|_H^2 + \frac{1}{22} \|\partial \phi_\lambda(\tau, \cdot) u_\lambda(\tau)\|_H^2 + \frac{1}{101} \|B(\tau(t), \cdot) u_\lambda(t)\|_H^2 \right) d\tau + \phi_\lambda(t, u_\lambda(t)) \leq 25 \|f(t)\|_H^2 + 3 \eta(t) (1 + \phi_+(t, u_\lambda(t))) + 3M \|u_\lambda(t)\|_H^2,
\]
for a.e. $t \in [0, T]$, whence by integrating and by $\phi_\lambda \leq \phi \leq \phi_+$ [Br, p. 39],
\[
\int_0^t \left(\frac{1}{303} \|u'_\lambda(\tau)\|_H^2 + \frac{1}{22} \|\partial \phi_\lambda(\tau, \cdot) u_\lambda(\tau)\|_H^2 + \frac{1}{101} \|B(\tau(t), \cdot) u_\lambda(t)\|_H^2 \right) d\tau + \phi_\lambda(t, u_\lambda(t)) \leq \phi(0, u_0) + 25 \|f\|_{L^2(0, T; H)} + 3 \|\eta\|_{L^1(0, T)} + 3M \int_0^t \|u_\lambda(\tau)\|_H^2 d\tau + \int_0^t 3 \eta(t) \frac{1}{2} (1 + \text{sgn} \phi_\lambda(\tau, u_\lambda(\tau))) \phi_\lambda(t, u_\lambda(\tau)) d\tau.
\]
By Gronwall’s inequality [Br, p. 156], there is a constant $M_5 > 0$ such that
\[
\phi_\lambda(t, u_\lambda(t)) \leq M_5 \left(1 + \phi_+(0, u_0) + \|f\|_{L^2(0,T;H)}^2 \int_0^t \|u_\lambda(\tau)\|_H^2 \, d\tau\right),
\]
for each $t \in [0, T]$. By (4.8) and by (4.9), for each $t \in [0, T]$,
\[
\|u_\lambda(t)\|_H^2 \leq 2\|u_0\|_H^2 + 606T \frac{1}{303} \int_0^t \|u'_\lambda(\tau)\|_H^2 \, d\tau \leq 2\|u_0\|_H^2 - 606T \phi_\lambda(t, u_\lambda(t))
\]
\[
+ M_6 \left(1 + \phi_+(0, u_0) + \|f\|_{L^2(0,T;H)}^2 \int_0^t \|u_\lambda(\tau)\|_H^2 \, d\tau\right).
\]
(4.10)

By the definition for subdifferential and by $\phi_\lambda(t, \cdot) \geq \phi(t, J_\lambda(t)\cdot)$ [Br, p. 39],
\[
\phi_\lambda(t, x) \geq \phi(t, J_\lambda(t)z(t)) + \langle \partial \phi_\lambda(t, \cdot)z(t), x - z(t) \rangle_H
\]
\[
\geq -\frac{1}{909T} \|x\|_H^2 + \phi(t, z(t)) - (228T + 1)\|\partial \phi^0(t, \cdot)z(t)\|_H^2 - \|z(t)\|_H^2
\]
for each $t \in [0, T]$ and $x \in H$. By (4.10), by (4.11), by integrating and by Gronwall’s inequality [Br, p. 156],
\[
\int_0^T \|u_\lambda(t)\|_H^2 \, dt \leq M_8 \left(1 + \|u_0\|_H^2 + \phi_+(0, u_0) + \|f\|_{L^2(0,T;H)}^2\right).
\]
(4.12)

Since $\phi(T, \cdot)$ is a proper convex lower semicontinuous function, it is bounded from below by an affine function [Br, p. 25]. Since there is $\tilde{z} \in D(\partial \phi(T, \cdot))$ and $\phi(T, J_\lambda(T)\cdot) \leq \phi_\lambda(T, \cdot)$, we have, for any $x \in H$,
\[
-\phi_\lambda(T, x) \leq M_8 \|J_\lambda(T)x\|_H + M_8 \leq \frac{1}{909T} \|x\|_H^2 + M_8.
\]
(4.13)

We obtain now Lemma 4.2 from (4.8)-(4.10) and from (4.12).

**Lemma 4.3.** There are a subsequence of $(\lambda)$ tending to $0+$, $v, w \in L^2(0,T;H)$, and $u \in H^1(0,T;H)$ such that $u'_\lambda \rightharpoonup u'$, $\partial \phi_\lambda u_\lambda \rightharpoonup v$, $B_\lambda u_\lambda \rightharpoonup w$ weakly in $L^2(0,T;H)$ and $u_\lambda \rightarrow u$ in $C([0,T];H)$, as $\lambda \rightarrow 0+$.

**Proof.** By Lemma 4.2, there are $u'^*, v, w \in L^2(0,T;H)$ and a subsequence with $u'_\lambda \rightharpoonup u'^*$, $\partial \phi_\lambda u_\lambda \rightarrow v$ and $B_\lambda u_\lambda \rightharpoonup w$ weakly in $L^2(0,T;H)$.

Let $\lambda, \mu \in [0,1]$. By (4.5a) and by the monotonicity of $\partial \phi_\lambda(t, \cdot)$ and of $B_\lambda(t)$,
\[
\frac{d}{dt} \frac{1}{2} \|u_\lambda(t) - u_\mu(t)\|_H^2 \leq (\lambda + \mu) \left(2\|\partial \phi_\lambda(t, \cdot)u_\lambda(t)\|_H^2 + 2\|\partial \phi_\mu(t, \cdot)u_\mu(t)\|_H^2\right) + \left(\lambda + \mu\right) \left(2\|B_\lambda(t)u_\lambda(t)\|_H^2 + 2\|B_\mu(t)u_\mu(t)\|_H^2\right),
\]
for a.e. $t \in [0,T]$,
(see [Br, p. 56]), whence by integrating, by (4.5b) and by Lemma 4.2,
\[ \|u_\lambda - u_n\|_{C([0,T];H)}^2 \leq 8(\lambda + \mu) \left( M_1 \|f\|_{L^2(0,T;H)}^2 + M_2 \left( 1 + \|u_0\|_H^2 + \phi(0, u_0) \right) \right). \]

Hence \((u_\lambda)\) converges toward some \(u \in C([0,T];H)\). Since the derivative is a strongly-weakly closed mapping in \(L^2(0,T;H)\), \(u' = u^*\). Thus \(u \in H^1(0,T;H)\).

By Lemmas 4.2 and 4.3 and by the definition of the Yosida approximate,
\[ \|u - J_\lambda u_\lambda\|_{L^2(0,T;H)} \leq \|u - u_\lambda\|_{L^2(0,T;H)} + \lambda \|\partial \phi u_\lambda\|_{L^2(0,T;H)} \to 0, \quad \text{as } \lambda \to 0+. \]

Since \(\partial \phi u_\lambda \in \partial \phi J_\lambda u_\lambda, \partial \phi u_\lambda \to v\) weakly in \(L^2(0,T;H)\) and \(\partial \phi\) is a maximal monotone operator, the demiclosedness result [Br, p. 27] implies \(v \in \partial \phi u\). Similarly, \(w \in Bu\). Hence \((u, v, w)\) is a solution for (4.1). Lemmas 4.2 and 4.3 and the weak lower semicontinuity of the norm of \(L^2(0,T;H)\) and of \(\phi(t, \cdot)\) imply (4.2). Indeed, for a subsequence,
\[ \phi(t, u(t)) \leq \liminf_{\lambda \to 0^+} \phi(t, J_\lambda(t) u_\lambda(t)) \leq \liminf_{\lambda \to 0^+} \phi_\lambda(t, u_\lambda(t)), \quad \text{for a.e. } t \in [0,T]. \]

Let \((u, v, w), (\tilde{u}, \tilde{v}, \tilde{w}) \in H^1(0,T;H) \times L^2(0,T;H)^2\) be two solutions of (4.1). Then by (4.1a) and by the monotonicity of \(\partial \phi(t, \cdot)\) and of \(B(t)\),
\[ \frac{d}{dt} \frac{1}{2} \|u(t) - \tilde{u}(t)\|_H^2 = (u(t) - \tilde{u}(t), -v(t) - w(t) + \tilde{v}(t) + \tilde{w}(t))_H \leq 0, \]
for a.e. \(t \in [0,T]\), whence by integrating over \([0,t] \subset [0,T]\) and by (4.1c), \(u = \tilde{u}\).

Theorem 4.1 is proved.

Proof of Theorem 4.2. Let \(u_{0n} \in D(\phi(0, \cdot))\) and \(f_n \in L^2(0,T;H)\),
\[ \|u_0 - u_{0n}\|_H \leq \frac{1}{n}, \quad f_n(t) = \begin{cases} f(t), & \text{if } \|f(t)\|_H \leq n, \\ 0, & \text{otherwise}, \end{cases} \]

By Theorem 4.1, there are \(u_n \in H^1(0,T;H), v_n, w_n \in L^2(0,T;H)\) satisfying
\[ u'_n(t) + v_n(t) + w_n(t) = f_n(t), \quad \text{for a.e. } t \in [0,T], \quad (4.14a) \]
\[ v_n(t) \in \partial \phi(t, \cdot) u_n(t), \quad w_n(t) \in B(t) u_n(t), \quad \text{for a.e. } t \in [0,T], \quad (4.14b) \]
\[ u_n(0) = u_{0n}. \quad (4.14c) \]

Lemma 4.4. There are some constants \(M_3, M_4 > 0\), independent on \(f\) and \(u_0\) and satisfying (4.3) with \((u_n, v_n, w_n)\) instead of \((u, v, w)\).

Proof. By the proof of Theorem 4.1 there are solutions \((u_{n\lambda})\) of (4.5) with \((u_{0n}, f_n)\) instead of \((u_0, f)\), satisfying Lemma 4.2 and converging in the sense of Lemma 4.3 toward \((u_n, v_n, w_n)\), a solution of (4.14).
By the definition for subdifferential, by the modified (4.5a), by \( \phi_\lambda \leq \phi \), and by the monotonicity of \( B_\lambda(t) \), for a.e. \( t \in [0, T] \),

\[
\phi_\lambda(t, u_{n\lambda}(t)) \leq (f_n(t) - B_\lambda(t)u_{n\lambda}(t) - u'_{n\lambda}(t), u_{n\lambda}(t) - z(t))_H + \phi_\lambda(t, z(t)) \\
\leq \left( \|f(t)\|_H + \|B^0(t)z(t)\|_H \right)\|u_{n\lambda}(t) - u_0\|_H - \frac{d}{dt} \frac{1}{2} \|u_{n\lambda}(t) - u_0\|_H^2 + \\
+ \frac{t}{2} \|f(t)\|_H^2 + \frac{t}{2} \|B^0(t)z(t)\|_H^2 + \frac{t}{606} \|u'_{n\lambda}(t)\|_H^2 + \frac{152}{t} \|u_0 - z(t)\|_H^2 + \phi(t, z(t)).
\]

We multiply the modified (4.7) by \( t \) and integrate it over \([0, t] \subset [0, T] \). Then

\[
\frac{1}{606} \int_0^t \tau \left( \|u'_{n\lambda}(\tau)\|_H^2 + \|\partial \phi_\lambda(\tau, \cdot)u_{n\lambda}(\tau)\|_H^2 + \|B_\lambda(t)u_{n\lambda}(\tau)\|_H^2 \right) \, d\tau + t\phi_\lambda(t, u_{n\lambda}(t)) + \\
+ \frac{1}{2} \|u_{n\lambda}(t) - u_0\|_H^2 \leq M_{11} \int_0^t (\|f(\tau)\|_H + \|u_{n\lambda}(\tau)\|_H)\|u_{n\lambda}(\tau) - u_0\|_H \, d\tau + \\
+ M_{11}(1 + M_f + M_{u_0}) + \int_0^t 3\eta(\tau) \frac{1}{2} \left( 1 + \text{sgn} \phi_\lambda(\tau, u_{n\lambda}(\tau)) \right) \tau \phi_\lambda(\tau, u_{n\lambda}(\tau)) \, d\tau, \quad (4.15)
\]

for each \( t \in [0, T] \). By Gronwall’s inequality \([Br, p. 156]\), for each \( t \in [0, T] \),

\[
t\phi_\lambda(t, u_{n\lambda}(t)) \leq M_{12} \left( 1 + M_f + M_{u_0} \right) + \\
+ M_{12} \int_0^t \left( \|f(\tau)\|_H + \|u_{n\lambda}(\tau)\|_H \right)\|u_{n\lambda}(\tau) - u_0\|_H \, d\tau. \quad (4.16)
\]

By the modified (4.5) and by the monotonicity of \( B_\lambda(t) \) and of \( \partial \phi_\lambda(t, \cdot) \),

\[
\frac{1}{2} \|u_{n\lambda}(t) - u_0\|_H^2 = \frac{1}{2} \|u_{0n} - u_0\|_H^2 + \int_0^t (u_{n\lambda}(\tau) - u_0, u'_{n\lambda}(\tau))_H \, d\tau \\
\leq \frac{1}{2} + \int_0^t \left( \frac{\tau}{2} \|u'_{n\lambda}(\tau)\|_H^2 + \frac{1}{\tau} \|z(\tau) - u_0\|_H^2 + \|f(\tau)\|_H \|u_{n\lambda}(\tau) - u_0\|_H + \\
+ \frac{t}{2} \|f(\tau)\|_H^2 - (u_{n\lambda}(\tau) - z(\tau), \partial \phi_\lambda(\tau, \cdot)z(\tau) + B_\lambda(\tau)z(\tau))_H \right) \, d\tau \\
\leq M_{13} + M_f + M_{u_0} + \int_0^T \frac{\tau}{2} \|u'_{n\lambda}(\tau)\|_H^2 \, d\tau + \int_0^t \tilde{\eta}(\tau)\|u_{n\lambda}(\tau) - u_0\|_H \, d\tau,
\]

where \( \tilde{\eta} \in L^1(0, 1) \). By the Gronwall type inequality, (4.15-16) and by (4.11),

\[
\|u_{n\lambda}(t)\|_H^2 \leq M_{14} + 4M_f + 4M_{u_0} + 2 \int_0^T \tau \|u'_{n\lambda}(\tau)\|_H^2 \, d\tau \\
\leq M_{15} + 4M_f + 4M_{u_0} + \frac{1}{2} \|u_{n\lambda}(T)\|_H^2, \text{ for each } t \in [0, T].
\]
By (4.16), (4.15) and by (4.11), \((u_{n\lambda}, \partial\phi_{\lambda}u_{n\lambda}, B_\lambda u_{n\lambda})\) satisfies (4.3). By the weak lower semicontinuity of the norms and by \(\liminf_{\lambda \to 0^+} \phi_{\lambda}(t, u_{n\lambda}(t)) \geq \phi(t, u_n(t))\), (4.3) is satisfied also by \((u_n, v_n, w_n, \phi(u_n))\).

**Lemma 4.5.** Denote \(u_n^*(t) = \sqrt{t} u_n(t), \hat{v}_n(t) = \sqrt{t} v_n(t)\) and \(\hat{w}_n(t) = \sqrt{t} w_n(t)\), for each \(t \in [0, T]\). There are \(u \in C([0, T]; H), u^*, \hat{v}, \hat{w} \in L^2(0, T; H)\) such that, on subsequences, as \(n \to \infty\),

\[
  u_n^* \to u^*, \quad \hat{v}_n \to \hat{v}, \quad \hat{w}_n \to \hat{w} \text{ weakly in } L^2(0, T; H) \text{ and } u_n \to u \text{ in } C([0, T]; H).
\]

**Proof.** The weak limits hold by the boundedness of \((\hat{u}_n^*), (\hat{v}_n)\) and of \((\hat{w}_n)\). By (4.5a) and by the monotonicity of \(\partial\phi(t, \cdot)\) and of \(B(t)\),

\[
  \frac{d}{dt} \frac{1}{2} \|u_n(t) - u_m(t)\|_H^2 \leq \|u_n(t) - u_m(t)\|_H \|f_n(t) - f_m(t)\|_H, \text{ for a.e. } t \in ]0, T[,
\]

whence by integrating, by (4.5b) and by the Gronwall type inequality,

\[
  \|u_n - u_m\|_{C([0,T]; H)} \leq \|u_0 - u_0\|_H + \int_0^T \|f_n(t) - f_m(t)\|_H dt.
\]

Thus \((u_n)\) is a Cauchy sequence, converging toward \(u \in C([0, T]; H)\).

**Lemma 4.6.** Denote \(v(t) = \hat{v}(t)/\sqrt{t}, \ w(t) = \hat{w}(t)/\sqrt{t}\), for each \(t \in ]0, T[\). Then \(u\) is differentiable and \(u'(t) = u^*(t)/\sqrt{t}, \ v(t) \in \partial\phi(t, \cdot)u(t), \ w(t) \in B(t)u(t)\) a.e. on \(]0, T[\).

**Proof.** Let \(\delta \in ]0, T[\) and \(\xi \in C^\infty_0(]0, T[)\). Then, as \(n \to \infty\),

\[
  \int_{\delta}^T u_n^*(t) \frac{\xi(t)}{\sqrt{t}} dt \leftarrow \int_{\delta}^T u_n^*(t) \frac{\xi(t)}{\sqrt{t}} dt - \int_{\delta}^T u_n(t) \xi'(t) dt \to - \int_{\delta}^T u(t) \xi'(t) dt.
\]

Thus \(u \in H^1(\delta, T; H)\) and \(u'(t) = u^*(t)/\sqrt{t}\), for a.e. \(t \in ]0, T[\). Since \(u_n \to u\) strongly and \(v_n \to v, \ w_n \to w\) weakly in \(L^2(\delta, T; H)\), the demiclosedness result of maximal monotone operators and Lemma 4.1 give that \(v \in \partial\phi u\) and \(w \in Bu\).

By the weak lower semicontinuity of the norms and of \(\phi(t, \cdot)\) and by Lemmas 4.4 and 4.5, \((u, v, w)\) satisfy (4.3) and (4.1). The uniqueness in \(u\) of the solution can be proved as in the proof of Theorem 4.1. Theorem 4.2 is proved.

**Proof of Theorems 4.3 and 4.4.** Assume (H.6). (Cf. [A, Prop. 4.2]). Define a complete metric space \(X\) with the norm \(\|\cdot\|_{C([0, T]; H)}\),

\[
  X = \{y \in C([0, T]; H) \mid y(t) \in D(C(t)), \text{ for each } t \in [0, T]\}.
\]

Define \(T : X \to X, \ Tx = u\), where \(u \in C([0, T]; H)\) is a solution of

\[
  \begin{align*}
  u'(t) + v(t) + w(t) &= f(t) - C(t)x(t), \text{ for a.e. } t \in [0, T], \quad (4.17a) \\
  v(t) &\in \partial\phi(t, \cdot)u(t), \ w(t) \in B(t)u(t), \text{ for a.e. } t \in [0, T], \quad (4.17b) \\
  u(0) &= u_0. \quad (4.17c)
  \end{align*}
\]
By Theorem 4.1 or 4.2, such \( u \in X \) exists and is unique, for each \( x \in X \).

Let \( x, y \in X \) and \( k = 1, 2, \ldots \). By (4.17), by (H.6) and by the monotonicities,

\[
\frac{d}{dt} \frac{1}{2} \| (Tx)(t) - (Ty)(t) \|_H^2 \leq \| (Tx)(t) - (Ty)(t) \|_H \eta(t) \| x(t) - y(t) \|_H,
\]

for a.e. \( t \in [0, T] \). By integrating, by (4.17c) and by the Gronwall type inequality,

\[
\| (Tx)(t) - (Ty)(t) \|_H \leq \int_0^t \eta(\tau) \| x(\tau) - y(\tau) \|_H d\tau, \text{ for each } t \in [0, T].
\]

Thus, by a classical argument,

\[
\| (T^k x)(t) - (T^k y)(t) \|_H \leq \frac{1}{k!} \| \eta \|^{k}_{L^1(0,T)} \| x - y \|_{C([0,T];H)}.
\]

Hence \( T^k \) is a strict contraction for \( k \) large enough. By Banach’s fixed point theorem \( T \) has a unique fixed point \( u \in X \). Thus (4.4) has a unique solution.

Next we do not assume (H.6). Denote \( M'_{u_0} = M_2(1 + \| u_0 \|^2_H + \phi(0,u_0)), M'_f = M_1 \| f \|_{L^2(0,T;H)}, M'_0 = 5 \| \eta \|_{L^1(0,T)} + 6M'_f + 12M'_{u_0}, \) and \( M'_1 = 48M_3 \| \eta \|^2_{L^1(0,T)} + 8 \| \eta \|_{L^1(0,T)} + 80M_3M_f + 6M_4M_{u_0} + M_4 \). Define, for \( \alpha = 0, 1, \)

\[
Y_{\alpha} = \left\{ y \in C([0,T];H) \mid \| y \|^2_{C([0,T];H)} + \int_0^T t^\alpha \psi(t, y(t)) dt \leq M'_\alpha \right\}.
\]

**Lemma 4.7.** Assume either the conditions of Theorem 4.3 \((\alpha = 0)\) or those of Theorem 4.4 \((\alpha = 1)\). In both cases, \( Y_{\alpha} \) is nonempty, convex and closed in \( C([0,T];H) \), \( T : Y_{\alpha} \to Y_{\alpha} \) is continuous and \( T Y_{\alpha} \) is compact in \( C([0,T];H) \).

**Proof.** By Theorem 4.1 or 4.2, (4.1) has a solution \( u \in C([0,T];H) \). By (4.2) or by (4.3) and by (H.8), \( u \in Y_{\alpha} \). By Fatou’s lemma and by the lower semicontinuity of \( \psi(\cdot) \), \( Y_{\alpha} \) is closed. The convexity of \( Y_{\alpha} \) is clear.

Assume the conditions of Theorem 4.3. Let \( x \in Y_0 \). Then \( Tx \in Y_0 \), since by (4.2) and by (H.8),

\[
\| Tx \|^2_{C([0,T];H)} + \int_0^T \psi(t, (Tx)(t)) dt \leq M_1 \| f - Cx \|^2_{L^2(0,T;H)} + M'_{u_0} + \| \eta \|^2_{L^1(0,T)} \\
\leq \frac{2M_1}{\beta} \int_0^T \psi(t, x(t)) dt + \frac{2M_1}{\beta} \| x \|^2_{C([0,T];H)} + (1 + \frac{2M_1}{\beta}) \| \eta \|_{L^1(0,T)} + 2M'_f + M'_{u_0} \leq M'_0.
\]

Let \( x, x_n \in Y_0, n = 1, 2, \ldots \), be such that \( x_n \to x \) in \( C([0,T];H) \), as \( n \to \infty \). Denote \( u_n = Tx_n \). By Theorem 4.1 there are \( v_n \) and \( w_n \) such that \((u_n, v_n, w_n)\) is a solution of (4.17) with \( x_n \) instead of \( x \). Moreover, by (4.2) and by (H.8),

\[
\int_0^T \left( \| u'_n(\tau) \|^2_H + \| v_n(\tau) \|^2_H + \| w_n(\tau) \|^2_H \right) d\tau \leq M'_0. \tag{4.18}
\]
Thus there are a subsequence and $u^*, v, w, c \in L^2(0, T; H)$ such that, as $n \to \infty$,
\[
u'_n \to u^*, \quad v_n \to v, \quad w_n \to w, \quad \text{and} \quad Cx_n \to c \text{ weakly in } L^2(0, T; H).
\]

By (H.7), there is $S \subset [0, T]$ of zero measure such that $\{u_n(t) \mid t \in [0, T] \setminus S, n = 1, 2, \ldots \}$ is relatively compact in $H$. By Ascoli’s theorem [D, p. 143] and by the equicontinuity of $\{u_n\}$, there is a subsequence and $\tilde{u} \in C([0, T] \setminus S; H)$ such that $u_n \to \tilde{u}$ in $C([0, T] \setminus S; H)$, as $n \to \infty$. Thus $u_n \to \tilde{u}$ in $L^2(0, T; H)$, as $n \to \infty$. As in the proof of Theorem 4.1, $\tilde{u}' = u^*, v \in \partial \phi \tilde{u}$ and $w \in B\tilde{u}$. By (H.9), $c = C\tilde{u}$. We extend $\tilde{u}$ to a function from $C([0, T]; H)$. By the uniqueness result of Theorem 4.1, $\tilde{u} = u = T x$.

Then,
\[
\|Tx_n - Tx\|_{C([0, T]; H)} = \|u_n - \tilde{u}\|_{C([0, T] \setminus S; H)} \to 0 \text{ as } n \to \infty.
\]

This limit holds for the whole sequence, since otherwise there were a subsequence and another solution for (4.17). Hence $T : Y_0 \to Y_0$ is continuous.

Let $y_n \in TY_0$, $n = 1, 2, \ldots$. Then $y_n$ satisfy (4.18). Again $(y_n)$ has a subsequence converging in $C([0, T]; H)$. Since $TY_0 \subset Y$ and $Y$ is closed, then $\overline{TY}_0$ is sequentially compact. Hence it is compact [KA, I.5.1].

Assume the conditions of Theorem 4.4. Let $x \in Y_1$. Then by (4.3) and (H.8),
\[
\|Tx\|_{L^\infty(0, T; H)}^2 + \int_0^T t \psi(t, (Tx)(t)) dt \leq \frac{13}{12} M_3 M_{C_x} + 4 M_3 M_f + M_4 M_{u_0} + M_4 \|\eta\|_{L^1(0, T)} \leq \frac{13 M_3}{4\beta} \left(\|x\|_{L^\infty(0, T; H)}^2 + \int_0^T t \psi(t, x(t)) dt\right) + \frac{3 M'_1}{16} \leq M'_1,
\]

where also $ab \leq \frac{1}{31} a^2 + 6 b^2$, (H.10), the Cauchy-Bunyakovsky inequality and $(a + b + c)^2 = 2a^2 + 3b^2 + 8c^2$ for $a, b, c \geq 0$, were used. Thus $Tx_1 \in Y_1$, i.e. $TY_1 \subset Y_1$.

Let $x_n, x \in Y_1$, $n = 1, 2, \ldots$, with $x_n \to x$ in $C([0, T]; H)$, as $n \to \infty$. Denote $u_n = Tx_n$. There are $v_n$ and $w_n$ satisfying (4.17) with $Cx_n$ instead of $Cx$, and
\[
\|u_n\|_{C([0, T]; H)} + \text{ess sup } \{t \phi_+(t, u_n(t)) \mid t \in [0, T]\} + \int_0^T t \left(\|u'_n(t)\|^2_H + \|v_n(t)\|^2_H + \|w_n(t)\|^2_H\right) dt \leq M'_1. \quad (4.19)
\]

The weighted space $L^2(0, T; H, t)$ is a Hilbert space. By (H.7), there are $t' \in [0, T]$, $u^*, v, w, c \in L^2(0, T; H, t)$, $u \in L^2(0, T; H)$, and $\tilde{u} \in H$ such that, for a subsequence,
\[
u'_n \to u^*, \quad v_n \to v, \quad w_n \to w, \quad Cx_n \to c \text{ weakly in } L^2(0, T; H, t),
\]
\[
u_n \to u \text{ weakly in } L^2(0, T; H) \text{ and } u_n(t') \to \tilde{u} \text{ in } H, \text{ as } n \to \infty.
\]
Let $\delta > 0$. Again $u^* = u'$ a.e. on $[\delta, T]$ and thus we may redefine $u$ on a set of zero measure such that $u \in H^1(\delta, T; H)$. Since $\delta$ arbitrary, $u \in C([0, T]; H)$. Indeed,

$$u(t) = \hat{u} + \int_{t'}^t u'(\tau) \, d\tau$$

and $u_n(t) \to u(t)$ weakly in $H$ on $[0, T]$, as $n \to \infty$. \hfill (4.20)

By (H.7) there is $S \subset [\delta, T]$ of zero measure such that $\{u_n\}$ is an equicontinuous family of mappings from $[\delta, T] \setminus S$ to a compact metric space. By Ascoli’s theorem [D, p. 143], there are a further subsequence $(n_k)$ and $\hat{u}$ such that

$$\|u_{n_k} - \hat{u}\|_{C([\delta, T] \setminus S; H)} \to 0, \text{ as } k \to \infty. \hfill (4.21)$$

However, by (4.20), $\hat{u}(t) = u(t)$, for each $t \in [\delta, T] \setminus S$. Thus (4.21) holds for the whole original subsequence. Since $u$ and $u_n$ are continuous on $[\delta, T]$,

$$\|u_n - u\|_{C([\delta, T]; H)} = \|u_n - u\|_{C([\delta, T] \setminus S; H)} \to 0, \text{ as } n \to \infty. \hfill (4.22)$$

By (4.17) and by the monotonicity of $\partial \phi(t, \cdot)$ and of $B(t)$,

$$\frac{d}{dt} \frac{1}{2}\|u_n(t) - u_0\|^2_H = (u_n(t) - u_0, f(t) - C(t)x_n(t) - v_n(t) - w_n(t))_H$$

$$\leq (1 + \frac{1}{4\epsilon})\eta_1(t) + \|u_n(t) - u_0\|_H \|C(t)x_n(t)\|_H + \eta_1(t) + \epsilon t\|v_n(t) + w_n(t)\|^2_H,$$

for a.e. $t \in [0, T]$ and for any $\epsilon > 0$; here $\eta_1, \eta_2 \in L^1(0, T)$ are independent on $\epsilon$ and on $n$. We integrate, use (4.19) and the Gronwall type inequality. Then, by (H.10) and by $x_n \in Y^1$, for each $t \in [0, T]$ and $\epsilon \in [0, 1[$,

$$\|u_n(t) - u_0\|_H \leq \left(4\epsilon M'_1 + 2(1 + \frac{1}{\epsilon}) \int_0^t \eta_1(\tau) \, d\tau \right)^{\frac{1}{2}} + \int_0^t \eta_1(\tau) \, d\tau + \int_0^t \|C(t)x_n(\tau)\|_H \, d\tau \leq 2\sqrt{\epsilon M'_1} + (1 + M'_1)\epsilon + \epsilon^{-2} \int_0^t \eta_2(\tau) \, d\tau. \hfill (4.23)$$

Define $u(0) = u_0$ and let $\delta \in [0, T], \epsilon \in [0, 1[$. Then by (4.20),

$$\|u_n - u\|_{C([0, T]; H)} \leq \sup_{0 < t \leq \delta} \|u_n(t) - u(t)\|_H + \sup_{\delta < t \leq T} \|u_n(t) - u(t)\|_H$$

$$\leq \sup_{0 < t \leq \delta} \left(\|u_n(t) - u_0\|_H + \lim_{m \to \infty} \|u_0 - u_m(t)\|_H\right) + \|u_n - u\|_{C([\delta, T]; H)}$$

$$\leq \sup_{0 < t \leq \delta} \sup_{m=1,2,...} 2\|u_m(t) - u_0\|_H + \|u_n - u\|_{C([\delta, T]; H)}.$$
Then, by (4.23) and by (4.22),
\[
\limsup_{n \to \infty} \|u_n - u\|_{C([0,T];H)} \leq 4\sqrt{\epsilon M_1^2 + 2\epsilon (1 + M_1')} + 2\epsilon^{-2} \int_0^\delta \eta_2(\tau) \, d\tau \\
\to 4\sqrt{\epsilon M_1^2 + 2\epsilon (1 + M_1')} \to 0, \text{ as } \delta \to 0, \epsilon \to 0, \text{ successively.}
\]

Hence \(u_n \to u\) in \(C([0,T];H)\), as \(n \to \infty\), and thus \(u \in C([0,T];H)\).

By the demiclosedness result, by Lemma 4.1 and by (H.9),
\[
v(t) \in \partial \phi(t, \cdot)u(t), \quad w(t) \in B(t)u(t), \quad c(t) = C(t)u(t), \text{ for a.e. } t \in ]\delta, T[
\]
and for each \(\delta \in ]0, T[\). Thus \(u, v, w\) satisfy (4.17). Since the solution of (4.17) is unique in \(u\), \(u_n \to u = T x\) in \(C([0,T];H)\), as \(n \to \infty\). Thus \(T\) is continuous.

Let \(u_n \in T Y_1, n = 1, 2, \ldots\). As above, there are a subsequence and \(u \in C([0,T];H)\) such that \(u_n \to u\) in \(C([0,T];H)\), as \(n \to \infty\). Since \(Y_1\) is closed and \(TY_1 \subset Y_1\), then \(TY_1\) is sequentially compact. Thus \(TY_1\) is compact.

Lemma 4.7 is proved.

By Schauder’s fixed point theorem [KA, XVI, 4.2], \(T : Y_\alpha \mapsto Y_\alpha\) has a fixed point \(u \in Y_\alpha, \alpha = 0, 1\). Theorems 4.3 and 4.4 are proved.

5. The case of time-dependent \(G\) and \(K\)

Let \(\delta, M, T > 0, H = L^2(0,1), V = H^1(0,1), \) and \(\eta \in L^1(0, \max(1,T))\). We state on \(g, k : [0,T] \times [0,1] \times \mathbb{R} \mapsto - \infty, \infty] \) and \(j : [0,T] \times \mathbb{R}^2 \mapsto - \infty, \infty]\), for each \(t \in [0,T]\) and for a.e. \(x \in ]0,1]\):

(J1) \(g(t, \cdot, \cdot)\) and \(k(t, \cdot, \cdot)\) are measurable with respect to the \(\sigma\)-field generated by the products of Lebesgue sets in \([0,T]\) and Borel sets in \(\mathbb{R}\).

(J2) \(g(t, x, \cdot)\) and \(k(t, x, \cdot)\) are proper convex lower semicontinuous functions.

(J3) \(j(t, \cdot)\) is a proper convex lower semicontinuous function.

Denote \(G(t, x) = \partial g(t, x, \cdot)\), \(K(t, x) = \partial k(t, x, \cdot)\) and \(\beta(t) = \partial j(t, \cdot)\).

(J4) \((\tilde{y}_1 - \tilde{y}_2)(y_1 - y_2) \geq \delta(y_1 - y_2)^2, \) for each \((y_1, \tilde{y}_1), (y_2, \tilde{y}_2) \in G(t, x)\).

(J5) There is \(z : [0,T] \mapsto V\) such that \(k(t, \cdot, z(t, \cdot)), g(t, \cdot, z(t, \cdot)) \in L^1(0,1), \) \(G(t, \cdot) z_z(t, \cdot), K(t, \cdot) z(t, \cdot) \in H, \) and \((z(t, 0), z(t, 1)) \in D(\beta(t))\).

(J6) For each \(m > 0\), there is \(\eta_{m,t} \in H\) such that \(|K(t,x)^0 y| \leq \eta_{m,t}(x), \) whenever \(y \in [-m, m]\).

(J7) Either \(\beta(t)\) is bounded or \(G(t,x)\) is bounded.

Remark 5.1. ([BP, p. 116]). Assume \(J_2\) and, for a.e. \(x \in ]0,1]\) and each \(t \in [0,T]\), \(\text{int} \, D(g(t, x, \cdot)) \neq \emptyset \neq \text{int} \, D(k(t, x, \cdot))\). Then \(J_1\) is equivalent to \(J_1'\).

(J1') For each \(t \in [0,T]\) and \(y \in \mathbb{R}\), \(g(t, \cdot, y)\) and \(k(t, \cdot, y)\) are measurable.

Define \(\phi : [0,T] \times H \mapsto ]-\infty, \infty]\),
\[
\phi(t, y) = \begin{cases} 
\frac{1}{\delta} \int_0^1 g(t, x, y'(x)) \, dx + \int_0^1 k(t, y(x)) \, dx + \\
+ j(t, y(0), y(1)), \text{ if } y \in H^1(0,1), \\
\infty, \text{ otherwise},
\end{cases} \tag{5.1}
\]
Lemma 5.1. Assume $J_1$-$J_5$ and let $t \in [0,T]$. Then $\phi(t, \cdot)$ is a proper convex lower semicontinuous function. If, in addition, $J_6$ and $J_7$ are satisfied, then $\partial \phi(t, \cdot) = A(t)$ and $u_\lambda(t) = (I + \lambda A(t))^{-1} y$ satisfies, for some constant $M_0 > 0$ and for any $y \in H$, $\lambda \in [0,1]$,

$$
\|u_\lambda(t)\|^2_H + \lambda \|u_\lambda(t)\|^2_V \leq M_0 \left( \|y\|^2_H + \|z(t)\|^2_H + \|K(t, \cdot)0^0(z(t))\|^2_H + \\
+ \lambda \|G(t, \cdot)0(z(t,0), z(t,1))\|^2_{R^2} \right). \tag{5.3}
$$

Let us state more hypotheses. Lemma 5.1 will be proved later.

(J8) There is $h_0 \in [0,T]$ such that for each $\lambda \in [0,1]$, $h \in [0,h_0[, t \in [0,T]$, $y \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^2$ and for a.e. $x \in [0,1]$,

$$(\bar{y}_1 - \bar{y}_2)(y_1 - y_2) \geq -\eta(t) h^2 \left( \eta(x) + y_1^2 + y_2^2 + g_+(t+h, y_1) + g_+(t, y_2) \right) + \\
+ \delta(y_1 - y_2)^2,$$

whenever $(y_1, \bar{y}_1) \in G(t+h, x), (y_2, \bar{y}_2) \in G(t,x)$,

$$
\|\beta_\lambda(t \pm t)\mathbf{y} - \beta_\lambda(t)\mathbf{y}\|_{R^2} \leq \eta(t) h^2 \left( 1 + \|y\|_{R^2} + \|\beta_\lambda(t)\mathbf{y}\|_{R^2} + j_\lambda(t, \mathbf{y}) \right),
$$

$$
\sup \{ |\xi| \mid \xi \in G(t,x) y \} \leq M|y| + \sqrt{\eta(x)}.
$$

(J9) $\text{ess sup} \left\{ \|K^0(t, \cdot)z(t)\|_H + \|z(t)\|_H + |\phi(t, z(t))| + \|\beta^0(t)(z(t,0), z(t,1))\|_{R^2} + \\
+ \|G^0(t, \cdot)z(t, \cdot)\|_H \mid t \in [0,T] \right\} < \infty$.

(J10) For each $h \in [0,h_0[, t \in [0,T-h[, y \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^2$, $\lambda \in [0,1]$, and for a.e. $x \in [0,1]$,

$$
|g(t+h,x,y) - g(t,x,y)| \leq h \sqrt{\eta(t)} \left( \eta(x) + y^2 + |G(t,x)0y| + g_+(t,x,y) \right),
$$

$$
|j_\lambda(t+h,y) - j_\lambda(t,y)| \leq \sqrt{\eta(t)} \left( 1 + \|y\|_{R^2} + \|\beta_\lambda(t)\mathbf{y}\|_{R^2} + j_\lambda(t,y) \right).
$$

Lemma 5.2. Assume $J_1$-$J_9$ and let $y \in H$, $\lambda \in [0,1]$, $g, j \geq 0$, and $k \equiv 0$. Then $u_\lambda \in H^1(0,T; V)$ and there is a constant $M^* > 0$, satisfying for a.e. $t \in [0,T]$,

$$
\|u_\lambda(t)\|^2_H + \lambda \|u_\lambda(t)\|^2_V \leq M^* \left( 1 + \|u_\lambda(t)\|^2_V + \phi(t,u_\lambda(t)) + \\
+ \|\phi_\lambda(t, \cdot)\|_H + \limsup_{h \to 0+} \int_0^1 g(t+h,x,u_\lambda(x,t)) \, dx \right). \tag{5.4}
$$

Proof. (Hint). We estimate $\int_0^{T-h} \|u_\lambda(t+h) - u_\lambda(t)\|^2_V \, dt$ using $u_\lambda + \lambda Au_\lambda \ni y$ and the approximate problems with $\beta_\mu(t)$ instead of $\beta(t)$ and their limits as $\mu \to 0+$, cf. the end of the proof of Lemma 5.1.
Lemma 5.3. Assume the conditions of Lemma 5.2 and \(J_{10}\). Let \(u_0 \in H\) and \(f \in L^1(0,T;H)\) be such that \(M_f + M_{u_0} < \infty\) and define \(B(t) \equiv 0\). Then all the conditions of Theorem 4.2 are satisfied.

**Proof.** (Hint). For (H.3) we estimate \(\int_0^{T-h} |\phi_{\lambda}(t + h, y) - \phi_{\lambda}(t, y)|^2 dt\).

By Theorem 4.2, (4.3), \(J_4\) and by \(J_9\), there are thus \(u \in C([0,T]; L^2(0,1))\), differentiable a.e. on \([0,T]\), and measurable \(w : [0,T] \mapsto H^1(0,1)\) satisfying (1.1) with \(K(t,x) \equiv 0\) and

\[
\text{ess sup}_{t \in [0,T]} \int_0^1 \left( u(t,x)^2 + w(t,x)^2 \right) dx + \int_0^T \int_0^1 \left( u(t,x)^2 + w(t,x)^2 \right) dx dt < \infty.
\]

**Proof of Lemma 5.1.** Since \(G(t,x)\) and \(K(t,x)\) are subdifferentials,

\[
g(t,x,y) \geq g(t,x,z_t(x)) + G(t,x)^0 z_t(x)(y - z_t(x)),
\]

\[
k(t,x,y) \geq k(t,x,z_t(x)) + K(t,x)^0 z(t,x)(y - z(t,x)).
\]

Thus \(g\) and \(k\) are normal convex integrands. Hence the integrals in (5.1) are well-defined, taking values from \([-\infty, \infty]\). The convexity of \(\phi(t,\cdot)\) is clear. In order to prove the lower semicontinuity, let \(\lambda > 0\) and consider the level sets \(S_\lambda = \{y \in H \mid \phi(t,y) \leq \lambda\}\). Let \(y_n \in S_\lambda\) and \(y \in H\), \(n = 1,2,\ldots\), be such that \(y_n \rightharpoonup y\) in \(H\), as \(n \to \infty\). By \(J_4, J_5\) and the definition for subdifferential we obtain, for a.e. \(x \in [0,1]\),

\[
g(t,x,y_n'(x)) \geq \frac{\delta}{8} y_n'(x)^2 + g(t,x,z_t(x)) - 2G(t,x)^0 z_t(x) - \delta z(t,x)^2.
\]

(5.6)

By (5.1) and by \(J_5\), \((y_n)\) is bounded in \(V\). So, for a subsequence, \(y_n \rightharpoonup y\) weakly in \(V\), as \(n \to \infty\). By Mazur’s lemma we can form convex combinations \(z_n\) of \(y_n\)’s such that \(z_n \rightharpoonup y\) strongly in \(V\). Thus \(z_n \in S_\lambda\) and for a subsequence,

\[
z_n(x) \to y(x),\text{ for each }x \in [0,1],\text{ and }z_n'(x) \to y'(x),\text{ for a.e. }x \in [0,1].
\]

By Fatou’s lemma we obtain now (cf. [BP, p. 117]) that \(y \in S_\lambda\), since

\[
\lambda \geq \liminf_{n \to \infty} \int_0^1 g(t,x,z_n'(x)) dx + \liminf_{n \to \infty} \int_0^1 k(t,x,z_n(x)) dx + \liminf_{n \to \infty} j(t,z_n(0),z_n(1)) \geq \phi(t,y).
\]

Hence \(\phi(t,\cdot)\) is lower semicontinuous. Thus \(\partial \phi(t,\cdot)\) exists; evidently it contains \(A(t)\). So \(A(t) \subset H \times H\) is monotone. Hence it is maximal, if \(R(I + A(t)) = H\).
Lemma 5.4. (Cf. [M1, p. 250]). Let \( \gamma \subset \mathbb{R}^2 \times \mathbb{R}^2 \) be maximal monotone and

\[
D(F) = \{ w \in H^2(0,1) \mid (w'(0),w'(1)) \in \gamma(w(0),-w(1)) \}, \quad (Fw)(x) = -w''(x).
\]

Then \( F \) is maximal monotone in \( H \).

**Proof.** Clearly, \( F \) is monotone. Let us prove that \( R(I+F) = H \), i.e.

\[
-w''(x) + w(x) = y(x), \quad \text{for a.e. } x \in ]0,1[,
\]

(5.7a)

\[
(w'(0),w'(1)) \in \gamma(w(0),-w(1)).
\]

(5.7b)

has a solution \( u \in H^2(0,1) \), for each \( y \in H \). The general solution of (5.7a) is

\[
w(x) = c_1 e^x + c_2 e^{-x} + w_1(x)
\]

where \( w_1 \) is some solution of (5.7a). Thus (5.7b) is satisfied if \( \gamma y + \hat{\gamma} y \ni y_1 \), where \( y_1 \in \mathbb{R}^2 \) depends only on \( w_1 \) and

\[
\hat{\gamma} = \frac{2}{c^2 - 1} \begin{pmatrix} 1 + e^2 & 2e \\ 2e & 1 + e^2 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ -e & -e^{-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} w_1(0) \\ -w_1(1) \end{pmatrix}.
\]

Since \( \hat{\gamma} \) is continuous, monotone and coercive, \( \gamma + \hat{\gamma} \) is surjective [Ba, p. 48], and thus \( \gamma y + \hat{\gamma} y \ni y_1 \) has a solution \( y \in \mathbb{R}^2 \). Lemma 5.4 is proved.

Let \( y \in H \) and \( y_n \subset C_0^\infty([0,1]) \), \( n = 1, 2, \ldots \), be such that \( y_n \rightharpoonup y \) in \( H \), as \( n \to \infty \). Consider the problems (5.8);

\[
-w_n''(x) + G(t,x)^{-1}w_n(x) + \frac{1}{n} w_n(x) = y_n'(x), \quad \text{for a.e. } x \in ]0,1[,
\]

(5.8a)

\[
(w_n'(0),w_n'(1)) \in \beta(t)^{-1}(w_n(0),-w_n(1)).
\]

(5.8b)

Choosing in Lemma 5.4 \( \gamma = \beta(t)^{-1} \) and denoting by \( \hat{F} \) the realization of \( G(t,x)^{-1} + 1/n \) to \( H \), we see that (5.8) is equivalent to \( Fw_n + \hat{F}w_n = y_n' \). Since \( F + \hat{F} \) is surjective [Ba, p. 48], (5.8) has a solution \( w_n \in H^2(0,1) \). Define

\[
u_n(x) = w_n'(0) + \int_0^x G(t,\sigma)^{-1} w_n(\sigma) \, d\sigma, \quad v_n(x) = u_n(x) + \frac{1}{n} \int_0^x w(\sigma) \, d\sigma,
\]

(5.9a)

for each \( x \in [0,1] \). Then \( u_n, v_n \in H^1(0,1) \) and

\[
u_n(x) - \nu_n'(x) = y_n(x), \quad \text{for each } x \in [0,1],
\]

(5.9b)

\[
w_n(x) \in G(t,x)u_n'(x), \quad \text{for a.e. } x \in ]0,1[,
\]

(5.9c)

\[
(w_n(0),-w_n(1)) \in \beta(t)(v_n(0),v_n(1)).
\]

(5.9d)

We multiply (5.9b) by \( v_n - z(t) \) and integrate over \( [0,1] \). By J2-J5 we obtain

\[
\|u_n\|^2_{H^1(0,1)} + \|v_n\|^2_{H^1(0,1)} + \|w_n'(\cdot)\|^2_H + \frac{1}{n} \|w_n\|^2_H \leq M^* < \infty.
\]

(5.10)
If $\beta(t)$ is bounded, then $(w_n(0), -w_n(1))$ is bounded. Assume that $G(t,x), x \in [0,1]$, are bounded and $(w_n(0), -w_n(1))$ is unbounded. Then, by (5.10), $w_n(x)$ is unbounded, for each $x \in [0,1]$. By Fatou’s lemma,

$$
\liminf_{n \to \infty} \int_0^1 u_n'(x)^2 \, dx \geq \int_0^1 \liminf_{n \to \infty} \left( G(t,x)^{-1} w_n(x) \right)^2 \, dx = \infty,
$$

which contradicts (5.10). Hence in any case $(w_n(0), -w_n(1))$ is bounded. Thus we have a subsequence and $u, w \in H^1(0,1)$ such that, as $n \to \infty$,

$$
v_n \to u, \quad w_n \to w \quad \text{in} \quad H, \quad \text{and} \quad u_n' \to u', \quad w_n' \to w', \quad \text{weakly in} \quad H,
$$

$$
v_n(0) \to u(0), \quad v_n(1) \to u(1), \quad w_n(0) \to w(0), \quad w_n(1) \to w(1).
$$

Since the realization of $G(t,x)$ in $H$ and $\beta(t)$ are maximal monotone,

$$
\begin{align*}
&u(x) - u'(x) = y(x), \quad \text{for a.e. } x \in ]0,1[, \quad (5.11a) \\
&w(x) \in G(t,x)u'(x), \quad \text{for a.e. } x \in ]0,1[, \quad (5.11b) \\
&(w(0), -w(1)) = \beta(t)(u(0), u(1)). \quad (5.11c)
\end{align*}
$$

Hence $\hat{A}(t) = A(t)$, given by (5.2) with $K(t,x) \equiv 0$, is maximal monotone.

Let $m > 0$ and $K(t)$, $K^m(t)$ be the maximal monotone realizations to $H$ of $K(t,x)$ and of $K^m(t,x)$, respectively; $K^m(t,x)$ is maximal monotone, bounded by $\eta_{m,t}(x)$ and equals to $K(t,x)$ on $[-m, m]$. Then $D(K^m(t)) = H$. By [Br, p. 36], $\hat{A}(t) + K^m(t)$ is maximal monotone. Let $y \in H$. Then there is $(u_m, v_m) \in \hat{A}(t)$ such that $u_m + v_m + K^m(t)u_m \ni y$. We multiply this by $u_m - z(t)$ and obtain $u_m$ to be bounded in $H^1(0,1)$, independently on $m$. We choose $m$ to be so big that $K^m(t)u_m$ equals $K(t)u_m$. Thus $R(I + A(t)) = H$.

From $u_\lambda(t) + \lambda A(t)u_\lambda(t) \ni y$ we obtain (5.3). Lemma 5.1 is proved.

**Remark 5.2.** Assume $J_1$-$J_2$ and $J_4$-$J_7$ and let $\beta(t) \subset \mathbb{R}^2 \times \mathbb{R}^2$ be maximal monotone. Then $A(t)$, given by (5.2), is maximal monotone.

Indeed, $A(t)$ is clearly monotone. For $R(I + A(t)) = H$ in the proof of Lemma 5.1 we did not need $\beta(t)$ to be a subdifferential.

**References**


