A Variational Approach to a Problem Arising in Capillarity Theory

Gheorghe Moroşanu*

Faculty of Mathematics, ''A. I. Caza'' University, Bd. Copou 11, 6600 Iaşi, Romania

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This paper studies a model for capillarity in circular tubes. The main result of this paper states the existence of a unique $C^1$ solution for this model. This solution is a minimum point of some functional $\Psi$.

1. INTRODUCTION

Consider the problem

\[(tG(x'(t)))' = tx(t), \quad 0 \leq t \leq 1, \quad (1)\]

\[x'(1) = \beta, \quad (2)\]

where $\beta$ is a positive constant and $G$ satisfies the following assumptions:

(a) $G: [0, \beta] \rightarrow \mathbb{R}$ is continuous, strictly increasing, and $G(0) = 0$.

Note that in the particular case $G(u) = \mu u(1 + u^2)^{-1/2}$, $\mu > 0$, problem (1), (2) represents a model for capillarity in circular tubes [1, 4]. See also [2, pp. 289–293].

Problem (1), (2) was solved by A. Corduneanu and G. Moroşanu [1, 4] under more restrictive assumptions by using a direct method. Our aim here is to derive the same result by a variational approach.

*E-mail address: gmoro@dragon.uaic.ro.

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As we shall see later, it is convenient to extend $G$ outside the interval $[0, \beta]$ by linear functions. For example, let $G_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$G_1(u) = \begin{cases} 
  u, & \text{for } u < 0, \\
  G(u), & \text{for } 0 \leq u \leq \beta, \\
  u - \beta + G(\beta), & \text{for } u > \beta.
\end{cases}$$

Obviously $G_1 \in C(\mathbb{R})$ and $G_1$ is strictly increasing. We shall prove that the equation

$$(tG_1(x'(t)))' = t\alpha(t), \quad 0 \leq t \leq 1$$

with boundary condition (2) has a unique solution $x \in C^4[0, 1]$ with $0 \leq x'(t) \leq \beta$ for $0 \leq t \leq 1$ and hence $x$ is also a solution of problem (1), (2).

Denoting $j(u) := \int_0^u G_1(s) \, ds$ we can expect to obtain the solution of (3), (2) in $C^4[0, 1]$ as a minimum point of the functional

$$\Psi(v) = \int_0^1 \left( j(v'(t)) + v^2(t)/2 \right) \, dt - G(\beta)v(1).$$

2. AUXILIARY RESULTS

Before stating and proving the main result we give two auxiliary results.

**Lemma 1.** If assumptions (A) are satisfied then for each $\beta > 0$, problem (3), (2) has at most one solution in $C^4[0, 1]$.

**Proof.** Let $x_1, x_2 \in C^4[0, 1]$ be solutions of (3), (2). We have

$$\int_0^1 (x_1 - x_2)(tG_1(x_1' - x_2')) \, dt = \int_0^1 t(x_1 - x_2)^2 \, dt.$$

Integrating by parts in the left hand side and using the monotonicity of $G_1$ and the fact that $x_1(1) = x_2(1) = \beta$ we can easily see that $x_1 = x_2$ in $[0, 1]$. Q.E.D.

**Lemma 2.** If assumptions (A) are satisfied, and $x \in C^4[0, 1]$ is a solution of Eq. (3), then the following implications hold:

$$x(0) = 0 \implies \text{either } x'(t) \geq 0 \text{ for } 0 \leq t \leq 1 \text{ or } x'(t) \leq 0 \text{ for } 0 \leq t \leq 1;$$

$$x(0) > 0 \implies x'(t) > 0 \quad \text{for } 0 < t \leq 1;$$

$$x(0) < 0 \implies x'(t) < 0 \quad \text{for } 0 < t \leq 1.$$
Proof. Multiplying \( x(t) \) by Eq. (3) and then integrating on \([0, t]\) we get

\[
\alpha(t)G_2(x'(t)) = \int_0^t s\{x^2(s) + x'(s)G_1(x'(s))\} \, ds, \quad 0 \leq t \leq 1. \tag{8}
\]

Therefore, taking into account the properties of \( G_1 \), we can see that

\[
\{t \in (0, 1); x(t) = 0\} = \{t \in (0, 1); x'(t) = 0\}
\]

and this set is either an empty set or an interval of the form \((0, \delta]\). Using this remark and Eq. (8) we can easily derive the conclusions (5), (6), and (7).

Q.E.D.

3. THE MAIN RESULT

**Theorem 1.** If assumptions (A) are satisfied then for each \( \beta > 0 \) problem (1), (2) has a unique solution in \( C^1[0, 1] \).

Proof. As uniqueness is already proved (see Lemma 1 which is also valid for problem (1), (2)) it remains to prove existence. To do that we shall use the functional \( \Psi \) defined by (4). Consider the space

\[
H = \{v = v(t); t^{1/2}v, t^{1/2}v' \in L^2(0, 1)\}.
\]

This is a Hilbert space with scalar product

\[
\langle v_1, v_2 \rangle_H = \int_0^1 t(v_1v_2 + v_1'v_2') \, dt.
\]

Now, since

\[
r^2/3 - C \leq j(r) \leq r^2 + C, \quad r \in \mathbb{R}, \tag{9}
\]

where \( C \) is a positive constant, we can see that \( \Psi \) is everywhere defined on \( H \) and moreover it is coercive,

\[
\Psi(v) \geq C_1\|v\|_H^2 - C_2, \quad v \in H, \tag{10}
\]

where \( C_1, C_2 \) are positive constants. Indeed for \( v \in H \) and \( \delta \) fixed in \((0, 1)\) we have

\[
|v(1)| \leq C_3(\|v\|_{L^2(\delta, 1)}^2 + \|v'\|_{L^2(\delta, 1)}^2)^{1/2}
\]

and so

\[
|v(1)| \leq C_4(\|t^{1/2}v\|_{L^2(\delta, 1)}^2 + \|t^{1/2}v'\|_{L^2(\delta, 1)}^2)^{1/2} \leq C_4\|v\|_H.
\]
This implies (10) by a straightforward computation. Since $\Psi$ is convex, continuous, and coercive it has a minimum point $x \in H$ (see, e.g., [3, p. 34]). Therefore we have (Euler–Lagrange equation)

$$(tG_1(x'(t)))' = tx(t), \quad \text{for a.e. } t \in (0, 1).$$

(11)

For any $\delta \in (0, 1)$ we can deduce from (11) and $x \in H$ that the function $f(t) := tG_1(x'(t))$ belongs to $C^1[\delta, 1]$ and so $x' \in C[\delta, 1]$ because $G_1^{-1} \in C(\mathbb{R})$. Hence $x \in C^1(0, 1)$. As $x$ is a minimum point of $\psi$ we also have

$$G_1(x'(1)) = G_1(\beta).$$

(12)

Hence $x$ verifies (2) and

$$tG_1(x'(t)) = G(\beta) - \int_t^1 sx(s) \, ds, \quad 0 < t \leq 1.$$  

(13)

From (13) we deduce that there exists

$$\lim_{t \to 0^+} tG_1(x'(t)) = l \in \mathbb{R}.$$  

By the definition of $G_1$ it follows that

$$l = \lim_{t \to 0^+} tx'(t).$$  

In fact $l = 0$ because otherwise

$$t^2(x'(t))^2 \geq l^2/2 > 0$$

in some interval $(0, \delta)$ and this contradicts the fact that $x \in H$. Now, it is easy to deduce from (11) that $x$ satisfies

$$tG_1(x'(t)) = \int_0^t sx(s) \, ds, \quad 0 \leq t \leq 1.$$  

(14)

In the next step we shall prove that $x \in C^1[0, 1]$. We have

$$\left| \int_0^t sx(s) \, ds \right| \leq 2^{-1/2} t \left( \int_0^t s^2(x(s)) \, ds \right)^{1/2}, \quad 0 \leq t \leq 1.$$  

Hence, by (14) we can see that $G_1(x'(t)) \to 0$ as $t \to 0^+$ and this implies $x'(t) \to 0$ as $t \to 0^+$. Therefore $x \in C^1[0, 1]$, $x'(0) = 0$ and $x$ verifies (3) for all $t \in [0, 1]$.

The final step is to prove that $x$ is a solution for Eq. (1). To this end we first note that $\beta > 0$ implies (see Lemma 2)

$$x'(t) \geq 0 \quad \text{and} \quad x(t) \geq 0 \text{ for } 0 \leq t \leq 1.$$  

(15)
Moreover, using (15) we can easily see that the function $t \mapsto (1/t) \int_0^t sx(s) \, ds$ is nondecreasing in $[0, 1]$. Since
\[
x'(t) = G_1^{-1} \left( \frac{1}{t} \int_0^t sx(s) \, ds \right)
\]
and $G_1^{-1}$ is nondecreasing it follows that $x'$ is also nondecreasing in $[0, 1]$. Therefore $0 = x'(0) \leq x'(t) \leq x'(1) = \beta$ for $0 \leq t \leq 1$ which implies that $x$ is a solution of Eq. (1).

Q.E.D.

4. FINAL COMMENTS

We have incidentally proved some properties of the solution of problem (1), (2) with physical significance (see [2–4] for the description of the model arising in capillarity theory): $x'(0) = 0$, (15), and the fact that $x'$ is nondecreasing (that is, $x$ is a convex function).

In order to point out some other properties of solutions let us assume in what follows that $G \in C(\mathbb{R})$, $G(0) = 0$, and $G$ is strictly increasing. By the above reasoning we can see that for any $\beta \in \mathbb{R}$ problem (1), (2) has a unique solution, say $x = x(t, \beta)$.

Now, if $x_1, x_2$ are solutions of Eq. (1) we have
\[
t[x_1(t) - x_2(t)] [G(x_1(t)) - G(x_2(t))] \\
= \int_0^t s \left[ (x_1 - x_2)^2 + (x_1' - x_2') [G(x_1') - G(x_2')] \right] \, ds, \quad 0 \leq t \leq 1.
\]

(17)

This implies that
\[
\{ t \in (0, 1) \, ; \, x_1(t) = x_2(t) \} = \{ t \in (0, 1) \, ; \, x_1'(t) = x_2'(t) \}
\]

(18)

and this set is either a void set or an interval $(0, \delta)$. Therefore, we have the monotonicity property
\[
\beta_1 < \beta_2 \Rightarrow x'(t, \beta_1) \leq x'(t, \beta_2) \quad \text{for} \, 0 \leq t \leq 1.
\]

(19)

Using (17), (18), and (19) we can easily deduce that
\[
\beta_1 < \beta_2 \Rightarrow x(t, \beta_1) \leq x(t, \beta_2) \quad \text{for} \, 0 \leq t \leq 1.
\]

(20)

As an immediate consequence of (19) and (20) we can show the continuity of the solution of $\beta$. More precisely, using Arzelà’s Criterion and Eq. (1) we can see that $\beta_n \to \beta$ implies $x(\cdot, \beta_n) \to x(\cdot, \beta)$ in $C^1[0, 1]$. 
On the other hand, if \( x_1, x_2 \) are solutions of Eq. (1) with \( x_1(0) < x_2(0) \) then \( x_1(t) < x_2(t) \) for \( 0 < t < 1 \) and \( x'_1(t) < x'_2(t) \) for \( 0 < t < 1 \) (see (18)).

Let us also mention that in the case in which we have uniqueness for the Cauchy problem associated to Eq. (1) (according to (16), this happens if \( G^{-1}_1 \) is locally Lipschitz), then making again use of (18) we can see that \( \beta_1 < \beta_2 \Rightarrow x'(t, \beta_1) < x'(t, \beta_2) \) for \( 0 < t < 1 \) and \( x(t, \beta_1) < x(t, \beta_2) \) for \( 0 < t < 1 \).

In particular, for \( \beta > 0 \), \( x'(t, \beta) > 0 \) in \( (0, 1] \) and \( x(t, \beta) > 0 \) in \( [0, 1] \). Moreover, in this case, it follows by (16) that \( x' \) is strictly increasing (i.e., \( x \) is strictly convex).

However, in general, that uniqueness property does not hold as the following simple counterexample shows. Let \( G \in C(\mathbb{R}) \) be a strictly increasing function such that \( G(u) = u^2/45 \) for \( 0 \leq u \leq 3 \). Then Eq. (1) with the initial condition \( x(0) = 0 \) has at least two solutions: \( x_1(t) = 0 \) and \( x_2(t) = t^3 \).

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