AN INTEGRAL-DIFFERENTIAL EQUATION FROM THE
CAPILLARITY THEORY

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The equation we are going to study in this Note appears on the base of capillarity of
continuous grain media.

Denoting by \( \theta_1, \theta_2, \theta_3 \) the Cartesian coordinates in the sense and choosing that the
base of grains \( H \) has its axis coincides with \( (Ox_2) \), we shall consider that the surfaces \( S \)
of the liquid in a capillarity surface whose parametric equations are

\[ x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z \]

where \( r \) and \( \varphi \) are the usual cylindrical coordinates. The surface \( S \) has a polar form of the
local equilibrium equation of Laplace [1, p. 131]

\[ \sigma = \frac{1}{2} \left( \kappa_1 \partial^2 \frac{1}{r} \right) + \frac{1}{2} \left( \kappa_2 \partial^2 \frac{1}{r} \right)
\]

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AN INTEGRO-DIFFERENTIAL EQUATION FROM THE CAPILLARITY THEORY

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The equation we are going to study in this Note appears in the case of capillarity in circular glass tubes.

Denoting by \((x_1, x_2, x_3)\) the Cartesian coordinates in the space and assuming that the tube of radius \(R > 0\) has its axis collinear with \((Ox_3)\), we shall consider that the surface \((S)\) of the liquid is a rotation surface whose parametric equations are

\[
(1) \quad x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = x_3(r) \quad \text{with} \quad 0 \leq r \leq R, \quad 0 \leq \varphi \leq 2\pi
\]

where \(r\) and \(\varphi\) are the usual cylindrical coordinates. The surface \((S)\) has a stable form if the local equilibrium equation of Laplace [1, p.111]

\[
(2) \quad \rho g x_3(r) = \sigma \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right), \quad 0 \leq r \leq R
\]

is satisfied at every its point. We use the notation \(\rho\) for the density of the liquid, \(g\) for the gravitational acceleration, \(R_1\) and \(R_2\) for the principal radii of curvature. The constant \(\sigma > 0\) is the interfacial tension liquid - air. Using the classical notations in the Differential Geometry for the coefficients of the two fundamental forms of a surface, we have in our case

\[
(3) \quad E = 1 + (x_3'(r))^2, \quad F = 0, \quad G = r^2
\]

and, respectively,

\[
(4) \quad L = \frac{r x_3''(r)}{\sqrt{EG - F^2}}, \quad M = 0, \quad N = \frac{r^2 x_3'(r)}{\sqrt{EG - F^2}}.
\]

Since the principal curvatures \(k_1\) and \(k_2\) satisfy

\[
(5) \quad k_1 + k_2 = \frac{EN + GL - 2FM}{EG - F^2},
\]

we obtain after some elementary computations...
\[
\frac{1}{R_1} + \frac{1}{R_2} = k_1 + k_2 = \frac{1}{r} \left[ \frac{rx_3'(r)}{\sqrt{1 + (x_3'(r))^2}} \right].
\]

From (2), integrating over the interval \([0, r]\), we obtain the integro-differential equation

\[
\sigma \frac{rx_3'(r)}{\sqrt{1 + (x_3'(r))^2}} = \rho g \int_0^r sx_3(s) ds, \quad 0 \leq r \leq R
\]

to which we associate the boundary condition \(x_3'(R) = \beta > o\). We remark that every solution also satisfies the condition \(x_3'(0) = o\). Making the substitution \(r = Rt\) (\(t\) = independent variable) and putting \(x_3(Rt) = Rx(t)\), we arrive to the problem

\[
\frac{\mu}{\sqrt{1 + (x'(t))^2}} = \frac{tx'(t)}{\int_0^t x(s) ds}, \quad 0 \leq t \leq 1; \quad x'(1) = \beta
\]

where \(\mu = \sigma / \rho g R^2 > o\) is a constant. We shall consider the problem (8) in a more general case, writing it under the form

\[
t G(x'(t)) = \int_0^t x(s) ds, \quad 0 \leq t \leq 1; \quad x'(1) = \beta > o
\]

and imposing conditions to the function \(G = G(u)\), which are satisfied in the particular case when \(G(u) = \mu u (I + u^2)^{-\frac{1}{2}}\), described by (8). We also consider the initial value problem

\[
t G(x'(t)) = \int_0^t x(s) ds, \quad 0 \leq t \leq 1; \quad x(0) = x_o.
\]

Our aim is to prove that, for every \(\beta > o\), there exists \(x_o = x_o(\beta) > o\), such that the corresponding solution of (10) satisfies (9), too.

Using the notation \(x' = y\), assuming that \(y\) is continuous and taking into account that

\[
x(t) = x_o + \int_0^t y(s) ds, \quad 0 \leq t \leq 1
\]

we shall obtain for the function \(y = y(t)\), from (10), the equation

\[
2t G(y(t)) = x_o t^2 + 2 \int_0^t s y(\tau) d\tau ds, \quad 0 \leq t \leq 1
\]

or its equivalent form

\[
2t G(y(t)) = x_o t^2 + \int_0^t (t^2 - s^2) y(s) ds, \quad 0 \leq t \leq 1.
\]
If $x = x(t)$ is a continuously differentiable solution of (10), then $y = x'$ satisfies equation (13). Conversely, if $y = y(t)$ is a continuous solution of (13), then $x = x(t)$ given by (11) satisfies equation (10). In what follows, we assume that the function $G$ satisfies the hypotheses

(H) $G = G(u)$ is defined for $u \geq 0$, $G = G(u)$ is continuously differentiable for $u \geq 0$, $G(0) = 0$, $G'(u) > 0$ for $u \geq 0$, $G'(u)$ is nonincreasing for $u \geq 0$ and $G(u) \to \mu$ $(\mu < \infty)$ as $u \to \infty$.

Let us denote, for $x_o > 0$,

(14) $K(x_o) = \{ y \in C_{[0,1]} : y(t) \geq 0, \int_0^t G(y(s)) ds \geq x_o t^2 + \int_0^t (t^2 - s^2) y(s) ds, 0 \leq t \leq 1 \}$

Proposition 1. The set of all points $x_o > 0$ for which $K(x_o) \neq \emptyset$ is an open interval $I = (0,A)$ with $0 < A < 2\mu$.

Proof. Let us first prove that, for every $M > 0$, we can choose a sufficiently small $x_o > 0$ such that $K(x_o)$ contains a certain function $y = y(t)$ with $0 \leq y(t) \leq M$ for $0 \leq t \leq 1$. Indeed, let us denote $a = G'(M)$ and let us consider the integral equation

(15) $2aty(t) = x_o t^2 + \int_0^t (t^2 - s^2) y(s) ds, 0 \leq t \leq 1$

Putting

(16) $u(t) = \int_0^t y(s) ds, 0 \leq t \leq 1$

it follows that $y = u'$. We obtain from (15) the differential equation

(17) $(atu'(t))' = x_o t + tu(t)$

and since $u(0) = 0$, $u'(0) = 0$ we search for $u = u(t)$ a series expansion of the form

(18) $u(t) = c_2 t^2 + c_3 t^3 + \cdots + c_n t^n + \cdots$

From (17) and (18), we obtain

(19) $u(t) = x_o \sum_{n=1}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2 a^n} = x_o I_0 \left( \frac{t}{\sqrt{a}} \right) - x_o$

where $I_0$ stands for the modified Bessel function of the first kind of order zero [2, p.181], [3, p.93], given by
(20) \[ I_o(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{t}{2} \right)^{2n}, \quad -\infty < t < \infty \]

For the solution \( y = y(t) \) of (15), we find

(21) \[ y(t) = u'(t) = \frac{x_o}{\sqrt{a}} I'_o \left( \frac{t}{\sqrt{a}} \right), \quad 0 \leq t \leq 1 \]

Now, it is clear that for a sufficiently small \( x_o > 0 \) we have \( 0 \leq y(t) \leq M \) if \( 0 \leq t \leq 1 \). 
Fixing now a point \( x_o \) with this property, we have for the corresponding solution \( y = y(t) \) of (15) that

(22) \[ G(y(t)) = G(o) + G' (\theta(t)) y(t) \quad \text{with} \quad 0 \leq \theta \leq y(t) \]

hence, taking into account that \( G(o) = o \) and \( G'(u) \) is nonincreasing for \( u \geq o \), it follows that

(23) \[ G(y(t)) \geq G'(y(t)) y(t) \geq G'(M) y(t) = a y(t), \quad 0 \leq t \leq 1 \]

We derive from (15) and (23) that \( y \in K(x_o) \), hence \( K(x_o) \neq \emptyset \). Since \( G < \mu \), it follows from definition (14) that \( K(x_o) = \emptyset \) for \( x_o \geq 2 \mu \). On another hand, \( o < x_o < x_o \) implies \( K(x_o) \subset K(x_o) \), hence \( K(x_o) \neq \emptyset \) implies \( K(x_o) \neq \emptyset \). Consequently, the set of points \( x_o > 0 \) with \( K(x_o) \neq \emptyset \) is an interval with the endpoints \( o \) and \( A, A < 2 \mu \). It remains to show that this interval (denote it by \( I \)) is open. To this end, it suffices to prove that \( x_o \in I \) implies \( K(x_o + \varepsilon) \neq \emptyset \) for a sufficiently small \( \varepsilon > 0 \). Fixing \( y \in K(x_o) \) and \( M > o \), we denote

(24) \[ M = M + \sup \{ y(t) : 0 \leq t \leq 1 \}, \quad a_t = G'(M) \]

If \( \alpha = \alpha(t) \) is the solution of the equation

(25) \[ 2 a_t t \alpha(t) = \varepsilon t^2 + \int_0^t (t^2 - s^2) \alpha(s) \, ds, \quad 0 \leq t \leq 1 \]

we know that, for a sufficiently small \( \varepsilon > 0 \), it follows \( o \leq \alpha(t) \leq M \) for \( 0 \leq t \leq 1 \). We deduce from

(26) \[ G(y(t) + o(t)) = G(y(t)) + G' (\theta(t)) o(t) \]

where \( y(t) \leq \theta(t) \leq y(t) + o(t) \) for \( 0 \leq t \leq 1 \), that

(27) \[ G(y(t) + o(t)) \geq G(y(t)) + G'(M) \alpha(t) = G(y(t)) + a_t \alpha(t) \]

also holds for \( t \in [o, 1] \). It follows from (27), (25) and (14) that \( y + \alpha \in K(x_o + \varepsilon) \), that is, \( K(x_o + \varepsilon) \neq \emptyset \). This concludes the proof.

Remark 1. Eq. (21) implies that from
(28) \[ o < x_o \leq B = \sup \left\{ \frac{M \sqrt{G'(M)}}{I_o'(1/\sqrt{G'(M)})} : M > o \right\} \]

it follows that the corresponding solution \( y = y(t) \) of (15) belongs to \( K(x_o) \), hence \( K(x_o) \neq \emptyset \). In other words, we have \( (o,B) \subset I \).

**Theorem 1.** For \( x_o \in I \), the integral equation (13) has a unique solution in \( C_{[o,I]} \).

**Proof.** Let \( y \) be a fixed element of \( K(x_o) \). We denote

\[ K(x_o, y) = \{ y \in K(x_o) : y(t) \leq y(t) \text{ for } o \leq t \leq I \} \]

Obviously \( y \in K(x_o, y) \), hence \( K(x_o, y) \neq \emptyset \). Moreover, \( K(x_o, y) \) is a closed subset of \( C_{[o,I]} \). We define an operator \( T : K(x_o, y) \rightarrow C_{[o,I]} \) by

\[ (Ty)(t) = G^{-1} \left( \frac{x_0 t}{2} + \int_o^t \left( t^2 - s^2 \right) y(s) \, ds \right), \quad o < t \leq I \quad ; \quad (Ty)(o) = o \]

where \( G^{-1} \) is the inverse function of \( G \), with \( \text{dom}(G^{-1}) = [o,\mu) \). We shall prove that

\[ T(K(x_o, y)) \subset K(x_o, y) \]

Indeed, if \( y \) is an element of \( K(x_o, y) \), we have for \( t \in (o,I) \)

\[ G(y(t)) \geq \frac{x_0 t}{2} + \int_o^t \left( t^2 - s^2 \right) y(s) \, ds \]

Applying the function \( G^{-1} \), we obtain \( y(t) \geq (Ty)(t) \) for \( t \in (o,I) \), but this is also true for \( t = o \). Hence, we have \( o \leq (Ty)(t) \leq y \) for \( t \in [o,I] \). Denoting \( Ty = z \), we obtain from (30), for \( t \in [o,I] \), that

\[ 2tG(z(t)) \geq x_0 t^2 + \int_o^t \left( t^2 - s^2 \right) y(s) \, ds \geq x_0 t^2 + \int_o^t \left( t^2 - s^2 \right) z(s) \, ds \]

Thus \( z = Ty \) satisfies both the conditions required by definition (29), hence the inclusion (31) is true.

If \( b > o \) is \( \sup \{ y(t) : o \leq t \leq I \} \), then for every \( y \in K(x_o, y) \), we have

\[ o \leq \frac{x_0 t}{2} + \int_o^t \left( t^2 - s^2 \right) y(s) \, ds \leq G(b) < \mu, \quad o < t \leq I \]

Denoting by \( L = L(b) = 1/G'(b) \) the Lipschitz constant of the function \( G^{-1} \) with respect to the interval \([o,G(b)]\), we can write

\[ |(Ty)(t) - (Tz)(t)| \leq \frac{L}{2t} \int_o^t \left( t^2 - s^2 \right) |y(s) - z(s)| \, ds, \quad o < t \leq I \]
for every pair of functions $y$ and $z$ of $K(x_o, \bar{y})$. Using the notation $\| \cdot \|$ for the norm in the space $C_{[o, l]}$, we obtain from the preceding inequality

$$
| (Ty)(t) - (Tz)(t) | \leq L \| y - z \| \frac{t^2}{3}, \quad 0 \leq t \leq l.
$$

By iterating this (upper bounding) inequality we find

$$
| (T^n y)(t) - (T^n z)(t) | \leq \frac{L^n \| y - z \|}{3^2 \cdot 5^2 \cdot \cdots \cdot (2n-1)^2} \frac{t^{2n}}{2n+1}, \quad 0 \leq t \leq l
$$

true for every $n \geq 2$. This fact shows us that the sequence of successive approximations \{ $T^n y_o$ \}_{n \geq o} \text{, where } y_o \text{ is an arbitrary element of } K(x_o, \bar{y}) \text{, is uniformly convergent on } [o, l] \text{ to a function } y \in K(x_o, \bar{y}) \text{, which is a fixed point of operator } T \text{ and a solution of equation (13).} \text{ We remark that this sequence is nonincreasing and tends to the solution } y = y(t) \text{ of (13) and we have } y(t) \leq \bar{y}(t) \text{ on } [o, l]. \text{ The unicity of the solution may be proved by a standard argument : if } z = z(t) \text{ is another solution of Eq.(13) in } C_{[o, l]} \text{ and we have } o \leq z(t) \leq b, \text{ for } o \leq t \leq l \text{ and if we denote } b_2 = \max \{ b_1, b \} \text{ and } L_2 = 1/G'(b_2) = \text{ the Lipschitz constant of the function } G^{-1} \text{ on the interval } [o, G(b_2)], \text{ we obtain}

$$
| y(t) - z(t) | \leq L_2 \| y - z \| \frac{t^2}{3}, \quad 0 \leq t \leq l
$$

Iterating this inequality, we deduce an (upper bounding) inequality similar to those given in (37). Hence $z(t) = y(t)$ and this completes the proof. 

**Remark 2.** Since the element $\bar{y}$ was taken (in the above proof) arbitrarily in $K(x_o)$, we have for the solution $y$ of (13)

$$
y(t) \leq \bar{y}(t) \text{ for } o \leq t \leq l, \quad \forall \bar{y} \in K(x_o).
$$

**Remark 3.** The following equivalence is true:

$$
x_o > o, \quad K(x_o) \neq \emptyset \iff \text{ equation (13) has a unique solution in } C_{[o, l]}.
$$

For $x_o \in I$, we denote by $y(t,x_o)$ the corresponding solution of equation (13).

**Proposition 2.** Assume that $x_o, \bar{x}_o \in I$. Then

(i) $y(t,x_o) > o$ for $0 < t \leq l$,

(ii) $y'(t,x_o) > o$ for $0 \leq t \leq l$,

(iii) $x_o < \bar{x}_o$ implies $y(t,x_o) < y(t,\bar{x}_o)$ for $0 \leq t \leq l$,

(iv) the mapping $x_o \mapsto y(t,x_o)$ is locally Lipschitzian from $I$ to $C_{[o, l]}$.

**Proof.** (i) It follows from Theorem 1 that $y(t,x_o) \geq o$ for $0 < t \leq l$. We remark that $y(o,x_o) = o$. Assume that there exists $t_1 > o$ such that $y(t_1,x_o) = o$. From the equation
we arrive to a contradiction since the last term is positive. Thus, \( y(t, x_o) > 0 \) for \( o < t \leq 1 \). (ii) Starting from

\[
y(t, x_o) = G^{-1}\left(\frac{x_o t}{2} + \frac{1}{2t} \int_0^t (t^2 - s^2) y(s, x_o) \, ds\right), \quad o < t \leq 1
\]

we easily deduce the existence and the continuity, for \( t \in (o, 1] \), of the derivative \( y'(t, x_o) = y_i'(t, x_o) \). From Eq. (13), we obtain

\[
G'(y(t, x_o))y'(t, x_o) = \frac{x_o}{2} + \int_0^t y(s, x_o) \, ds - \frac{1}{2t^2} \int_0^t (t^2 - s^2) y(s, x_o) \, ds
\]

also for \( t \in (o, 1] \). If we denote

\[
g(t) = 2t^2 \int_0^t y(s, x_o) \, ds - \int_0^t (t^2 - s^2) y(s, x_o) \, ds, \quad o \leq t \leq 1
\]

we have \( g(t) > 0 \) for \( t \in (o, 1] \), because \( g(o) = o \) and \( g'(t) > 0 \) for \( t \in (o, 1] \). Using (43) and (44), we can now write

\[
y'(t, x_o) > \frac{x_o}{2G'(y(t, x_o))} \geq \frac{x_o}{2G'(o)} > 0, \quad o < t \leq 1
\]

Making \( t \rightarrow o_+ \) in (43), we have

\[
y'(o, x_o) = \lim_{t \rightarrow o_+} y'(t, x_o) = \frac{x_o}{2G'(o)}.
\]

(iii) Denoting \( u(t) = y(t, x_o) - y(t, x_o) \), we have \( u(o) = o \) and \( u'(o) = (x_o - x_o)/2G'(o) > 0 \). This implies \( u(t) > 0 \) for \( t \) sufficiently small. Assume that \( u(t) > 0 \) on \((o, t_i)\) and \( u(t_i) = 0 \). We obtain from (13)

\[
o = (x_o - x_o)t_i^2 + \int_0^t (t_i^2 - s^2) u(s) \, ds,
\]

what is impossible. Thus \( u(t) > 0 \) on \((o, 1]\), that is, \( y(t, x_o) < y(t, x_o) \) on \((o, 1]\).

(iv) Let \([c, d]\) be a compact interval enclosed in \( I \). For every \( x_o \in [c, d] \) and \( t \in [o, 1] \) we have \( o \leq y(t, x_o) \leq y(1, d) \). Then we get from (42)

\[
o \leq \frac{x_o t}{2} + \frac{1}{2t} \int_0^t (t^2 - s^2) y(s, x_o) \, ds \leq G(y(1, d)) < \mu, \quad o < t \leq 1
\]

Taking \( x_o, \overline{x}_o \in [c, d] \), keeping for \( u = u(t) \) the same meaning as above, denoting by \( L' \) the Lipschitz constant of \( G^{-1} \) on the interval \([o, G(y(1,d))]) \) and making use of (42), we obtain
\( u(t) \leq \frac{L^*}{2} \left( \left| \bar{x}_o - x_o \right| + \int_0^t u(s) \, ds \right) , \quad 0 \leq t \leq 1 \).

The Gronwall-Bellman inequality implies
\( |y(t, \bar{x}_o) - y(t, x_o)| \leq \frac{L^*}{2} \left| \bar{x}_o - x_o \right| e^{\frac{L^*}{2}}, \quad 0 \leq t \leq 1 \)
what concludes the proof.

**Remark 4.** The function \( y = y(t, x_o) \) is continuous for \( t \in [0,1], x_o \in I \). Moreover, it may be proved the existence of the partial derivative \( \frac{\partial y}{\partial x_o}(t, x_o) = \eta(t, x_o) \), which satisfies the equation
\( G'(y(t, x_o)) \eta(t, x_o) = \frac{t}{2} + \frac{L^*}{2t} \int_0^t (t^2 - s^2) \eta(s, x_o) \, ds , \quad 0 < t \leq 1 \).

**Remark 5.** For a fixed \( t \in (0,1] \), the image of \( I \) by the function \( y = y(t, x_o) \) is an interval enclosed in \( (0, \infty) \) with its left endpoint \( = 0 \). In what follows, our aim is to prove that the image of \( I \) by \( y = y(1, x_o) \) is the whole halfaxis \( (0, \infty) \).

We again consider the main problem of this paper, namely: to find a function \( x = x(t) \) satisfying the conditions
\( t G(x'(t)) = \int_0^t s x(s) \, ds , \quad 0 \leq t \leq 1 ; \quad x'(1) = \beta > 0 \).

**Theorem 2.** Under the hypothesis (H), for every \( \beta > 0 \), the problem (52) has a unique solution in \( C^1_{[0,1]} \).

**Proof.** We recall that, with notation \( y = x' \), the initial value problem (10) may be reduced to the integral equation (13). For solving the problem (52), it must be proved the existence of \( x_o \in I \) such that the corresponding solution of (13) satisfies condition \( y(1, x_o) = \beta \). We know that the set \( \{ y(1, x_o) : x_o \in I \} \) is an interval with its left endpoint \( = 0 \). To complete the proof, it suffices to prove that \( \{ y(1, x_o) : x_o \in I \} = (0, \infty) \). Indeed, let us suppose that there exists a constant \( C > 0 \) such that \( y(t, x_o) \leq C \) for every \( x_o \in I \). Then we have \( 0 \leq y(t, x_o) \leq C \) for \( t \in [0,1] \) and \( x_o \in I \). In other words, the set \( \{ y(t, x_o) : x_o \in I \} \) of all the solutions of (13) is uniformly bounded on the interval \([0,1]\). On another hand, we deduce from Proposition 2 and from (43) that
\( o < y'(t, x_o) \leq \frac{A + C}{G'(C)} , \quad 0 \leq t \leq 1 , \quad x_o \in I \),
that is, the set of the derivatives is also uniformly bounded on \([0,1]\). The Arzelà-Ascoli theorem
implies that the function set \( \{ y(t, x_0^n) : x_0 \in I \} \) is relatively compact in \( C_{[o, I]} \). Now, let \( (x_0^n)_{n \geq 1} \) be an increasing sequence, convergent to \( A \). The sequence of functions \( (y(t, x_0^n)) \) is increasing and pointwise convergent on \([o, I]\) to a function \( y = \bar{y}(t) \), with \( o \leq \bar{y}(t) \leq C \) on \([o, I]\). But this convergence is uniform and it therefore follows that \( y = \bar{y}(t) \) is continuous on \([o, I]\). Making \( n \to \infty \) in the equation

\[
2tG(y(t, x_0^n)) = x_0^n t^2 + \int_0^t (t^2 - s^2) y(s, x_0^n) \, ds , \quad o \leq t \leq I , \quad n \geq 1
\]

we obtain

\[
2tG(\bar{y}(t)) = At^2 + \int_0^t (t^2 - s^2) \bar{y}(s) \, ds , \quad o \leq t \leq I ,
\]

what implies \( K(A) \neq \emptyset \), that is, \( A \in I \). This contradiction shows that the assumption that \( y(l, x_0) \leq C \) for every \( x_0 \in I \) is false. We conclude this proof with the remark that the mapping \( x_0 \to y(l, x_0) \) is an increasing bijection from \( I = (o, A) \) to \((o, \infty)\).

Remark 6. In connection with the problem (52), let us still notice that its solution \( x = x(t) \) also satisfies the additional condition \( x'(o) = o \) (i.e., \( y(o, x_0) = o \)) which has — by the way — a physical meaning.

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