ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS ASSOCIATED TO MONOTONE OPERATORS

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1. INTRODUCTION

Throughout this paper $H$ is a real Hilbert space with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. Consider the initial value problem:

$$\frac{du}{dt}(t) + Au(t) = f(t), \quad t > 0,$$  \hspace{1cm} (1.1)

$$u(0) = u_0,$$  \hspace{1cm} (1.2)

where:

$$A \text{ is a maximal monotone set in } H \times H,$$  \hspace{1cm} (1.3)

$$u_0 \in D(A).$$  \hspace{1cm} (1.4)

We will impose one of the following two conditions on $f$:

$$f \in L^1(0, \infty; H)$$  \hspace{1cm} (1.5)

$$f \in L^2(0, \infty; H).$$  \hspace{1cm} (1.6)

We suppose familiarity with the basic notions, notations and results concerning the monotone sets and Cauchy problem (1.1), (1.2) (for background information see [1, 2]). Our objective is to study the asymptotic behaviour, as $t \to \infty$, of the solution of (1.1), (1.2). So we generalize several results concerning the special case $f \equiv 0$ stated by Baillon and Brézis [3], Bruck [4] and Pazy [5]. We extend our results to some second order differential equations. The last section contains adequate examples. For more details and other aspects of this subject we refer the reader to [6–13].

2. WEAK CONVERGENCE OF SOLUTIONS

Lemma 2.1. Assume (1.3), (1.4), (1.5) hold and let $u$ be the unique integral solution of (1.1), (1.2). Then, $u(t)$ is bounded on $[0, \infty[$ if and only if $A^{-1}0$ is nonempty.

Proof. First, we suppose that $u(t)$ is bounded on $[0, \infty[$. Since $u(t)$ is integral solution of (1.1),

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(1.2) we have:
\[ 2^{-1}|u(t) - v|^2 - 2^{-1}|u_0 - v|^2 \leq \int_0^t (f(s) - w, u(s) - v) \, ds, \]
for every \( t \geq 0 \), \([v, w] \in A\).  \hfill (2.1)

Dividing by \( t > 0 \), we obtain:
\[
(2t^{-1}|u(t) - v|^2 - (2t^{-1}|u_0 - v|^2 + t^{-1}\int_0^t (f(s), v - u(s)) \, ds \leq (w, v - \sigma(t)),
\]
for every \( t > 0 \), \([v, w] \in A\), \hfill (2.2)

where
\[
\sigma(t) = t^{-1} \int_0^t u(s) \, ds
\]
is bounded on \([0, \infty[\), therefore there exists a sequence \( t_n \to \infty \) such that \( \sigma(t_n) \) converges weakly to an element \( p \in H \). If we take \( t = t_n \) in (2.2) and pass to limit it follows:
\[
0 \leq (w, v - p), \quad \text{for every} \quad [v, w] \in A.
\]

The last inequality and maximality of \( A \) implies that \([p, 0] \in A\). Conversely, if we suppose that \( A^{-1}0 \) is nonempty we can take \( v \in A^{-1}0 \), \( w = 0 \) in (2.1) and we deduce by a variant of Gronwall’s lemma (see [2], p. 157) that \( u(t) \) is bounded on \([0, \infty[\). Q.E.D.

We note that for \( f = 0 \) this lemma is due to Crandall and Pazy [7]. The next result generalizes a theorem stated by Baillon and Brézis [3]. For its proof we use a similar technique.

**Theorem 2.1.** Assume (1.3), (1.4), (1.5) hold and let \( u \) be the solution of (1.1), (1.2). If \( F = A^{-1}0 \) is nonempty, then
\[
\sigma(t) = t^{-1} \int_0^t u(s) \, ds
\]
converges weakly to an element \( p \in F \) and there exists
\[
\lim_{t \to \infty} ||\text{Proj}_F u(t) - p|| = 0.
\]

**Proof.** From Lemma 2.1 it follows that \( u(t) \) and \( \sigma(t) \) are bounded on \([0, \infty[\). Since \( u \) is an integral solution of (1.1), (1.2) we have:
\[
2^{-1}|u(t) - x|^2 - 2^{-1}|u_0 - x|^2 \leq \int_s^t |f(\tau)| \, |u(\tau) - x| \, d\tau,
\]
for every \( x \in F \), \( 0 \leq s \leq t < \infty \).

By the cited variant of Gronwall’s lemma this implies the following inequality:
\[
|u(t) - x| - |u(s) - x| \leq \int_s^t |f(\tau)| \, d\tau,
\]
for every \( x \in F \), \( 0 \leq s \leq t < \infty \).

Hence, for every \( x \in F \), the function \( t \to |u(t) - x| - \int_s^t |f(\tau)| \, d\tau \) is nonincreasing on \([0, \infty[\). Taking into account that this function is at the same time bounded on \([0, \infty[\), it follows that it has a finite limit, as \( t \to \infty \), for every \( x \in F \). Finally, since \( f \in L^2(0, \infty; H) \) we conclude that there exists
\[
\lim_{t \to \infty} |u(t) - x| = \rho(x), \quad \text{for every} \quad x \in F.
\]

We set:
\[
\nu(t) = \text{Proj}_F u(t).
\]

We shall prove that the function
\[
t \to |u(t) - \nu(t)| - \int_0^t |f(s)| \, ds
\]
is nonincreasing on \([0, \infty[\) and therefore \( |u(t) - \nu(t)| \) has limit as \( t \to \infty \). For fixed \( t \) we denote \( y(h) = u(t + h), \quad h \geq 0 \).

Then, \( y \) satisfies the following problem:
\[
dy \in \frac{d}{dh} y(h) + Ay(h) \ni f(t + h), \quad \text{a.e.} \quad h > 0; \quad y(0) = u(t).
\]

By the same argument above we obtain that the function
\[
h \to |y(h) - \nu(t)| - \int_0^h |f(h + s)| \, ds
\]
is nonincreasing and hence
\[
|u(t + h) - \nu(t)| - \int_0^{t + h} |f(s)| \, ds \leq |u(t) - \nu(t)|, \quad t, h \geq 0.
\]

This implies:
\[
|u(t + h) - \nu(t + h)| - \int_0^{t + h} |f(s)| \, ds \leq |u(t + h) - u(t)| - \int_0^t |f(s)| \, ds
\]
\[
- \int_0^{t + h} |f(s)| \, ds \leq |u(t) - \nu(t)| - \int_0^t |f(s)| \, ds,
\]
for every \( t \geq 0 \), \( h \geq 0 \).

Next, the procedure used in [3] is applicable with minor changes.

Now, we shall give a generalization of the principal result in [5, Th. 2.1].

**Theorem 2.2.** Assume (1.3), (1.4), (1.5) hold and let \( u \) be the solution of (1.1), (1.2) on \([0, \infty[\). Then, there exists the weak limit of \( u(t) \), as \( t \to \infty \) if and only if \( A^{-1}0 \) is nonempty and \( \omega_u \subset A^{-1}0 \), where \( \omega_u \) is the set of the weak cluster points of \([u(t)], t \geq 0 \).

**Proof.** “Only if” part. Suppose that \( u(t) \) converges weakly to an element \( p \in H \), as \( t \to \infty \). This implies that \( \sigma(t) \) converges weakly to \( p \). From (2.2) it follows that \( p \in F = A^{-1}0 \).
"If" part. Since $F$ is nonempty, according to Lemma 2.1, $\omega_u$ is nonempty too. Let $p, q$ be arbitrary in $\omega_u \subset F$. We have:
\[
|u(t) - p|^2 = |u(t) - q|^2 + 2(u(t) - q, q - p) + |q - p|^2, \quad t \geq 0
\]
and from (2.3) we obtain
\[
\rho^2(p) - \rho^2(q) = |q - p|^2.
\]
The same argument works with $p$ and $q$ reversed, i.e.,
\[
\rho^2(q) - \rho^2(p) = |p - q|^2.
\]
We conclude that $p = q$, hence $\omega_u$ contains only one element. The argument for "If" part was suggested by [13].

Q.E.D.

Theorem 2.3. Assume (1.4), (1.5) hold, $A = \partial \phi(\varphi: H \to \varnothing)$ is lower-semicontinuous and proper convex function, $A^{-1}$ is nonempty and let $u$ be the solution of (1.1), (1.2). Then, $u(t)$ converges weakly, as $t \to \infty$, to a point of $A^{-1} 0$.

If in particular $f = 0$, then this theorem is due to Bruck [4].

Proof. Taking into account Theorem 2.3 it remains to prove that $\omega_u \subset F = A^{-1} 0$. Let $\{t_n\} \subset [0, \infty]$ be an arbitrary sequence such that $t_n \to \infty$ and define the following functions:
\[
f_n(t) = \begin{cases} f(t) & \text{a.e. } t \in [0, 2^{-1} t_n] \\ 0 & \text{for every } t \in [2^{-1} t_n, \infty]. \end{cases}
\]

Consider the following problems:
\[
\frac{du}{dt}(t) + Au(t) = f_n(t), \quad t > 0 \tag{2.4}
\]
\[
u(n)(0) = u_0. \tag{2.5}
\]
Obviously, $u_n(t) = u(t), 0 \leq t \leq \frac{1}{2} t_n$ where $u_0$ is the solution of (2.4), (2.5). We set:
\[
y_n(t) = u(t + 2^{-1} t_n). \tag{2.6}
\]
Then, $y_n(t)$ verifies the following problem:
\[
\frac{d^+ y_n}{dt}(t) + A^0 y_n(t) = 0, \quad \text{for every } t > 0 \tag{2.7}
\]
\[
y_n(0) = u(2^{-1} t_n) \in D(\phi). \tag{2.8}
\]
We remember that $D(\phi) = \{x \in H; \phi(x) < \infty\}$ and $D(\phi) = D(A); A^0$ is the minimal section of $A$. It is also well known that if $u_n \in D(\phi)$ and $f \equiv 0$ on $[0, \infty[$, then (1.1), (1.2) has a unique strong solution $v$ on $[0, \infty]$ such that $v(t) \in D(A)$ for every $t > 0$, $v(t)$ is everywhere differentiable from the right for $t > 0$ and satisfies:
\[
\frac{d^+ v}{dt}(t) + A^0 v(t) = 0, \quad \text{for every } t > 0; \tag{2.9}
\]
the following estimation is also satisfied:
\[
\frac{d^+ v}{dt}(t) \leq |A^0 x| + t^{-1} |x - u_0|, \tag{2.10}
\]
for every $t > 0$, and $x \in D(A)$.

Since $F$ is nonempty $u(t)$ is bounded on $[0, \infty[$ (cf. Lemma 2.1). From (2.6) and (2.7) one obtains:
\[
|y_n(t) - x_n| \leq |u(2^{-1} t_n) - x_n|, \quad t > 0, \quad x_n \in F. \tag{2.11}
\]
By (2.8) it follows:
\[
\frac{d^+ y_n}{dt}(t) \leq t^{-1} |x_n - u(2^{-1} t_n)|, \quad t > 0, \quad x_n \in F. \tag{2.12}
\]
Hence, $\{t_n\}$ has a subsequence $\{t_{n_k}\}$ such that
\[
u(n_k(t_{n_k}) = y_n(2^{-1} t_{n_k}) \tag{2.13}
\]
converges weakly, as $k \to \infty$, to an element $p \in H$. By means of (2.10) and (2.6) one gets that $A^0 u_{n_k}(t_{n_k})$ converges strongly to $0$, as $k \to \infty$. Since $A$ is demiclosed it follows that $\{p, 0\} \in A$ and therefore $p \in F$. On the other hand, $u$ and $u_n$ being the integral solutions of (1.1), (1.2) and respectively (2.4), (2.5) one gets:
\[
|u(t) - u_n(t)| \leq \int_{2^{-1} t_n}^t |f(s)| ds, \quad t > 2^{-1} t_n. \tag{2.14}
\]
Thus
\[
\lim_{k \to \infty} |u_{n_k}(t_{n_k}) - u_{n_k}(t_{n_k})| = 0. \tag{2.15}
\]
Therefore $u_{n_k}$ converges weakly to $p \in F$. We conclude that $\omega_u \subset F$, as claimed.

Remark 2.1. Assume (1.3), (1.4), (1.5) hold and $F = \varnothing$. Then, it is clear by Lemma 2.1 that
\[
\lim_{t \to \infty} |u(t)| = \infty. \tag{2.16}
\]
For $A = \partial \phi$ and $f \equiv 0$, Pazy [5] proved that if $F = \varnothing$ or equivalently $0 \notin R(A)$, but $0 \in R(A)$, then
\[
\lim_{t \to \infty} |u(t)| = \infty. \tag{2.17}
\]
This fact remains true even if $f$ is an arbitrary function in $L(0, \infty; H)$. More precisely, we have:

Theorem 2.4. Assume (1.4), (1.5) hold, $A = \partial \phi, \phi(0) \in R(A)$ and let $u$ be the solution of (1.1), (1.2). Then, (2.11) is satisfied.

Proof. Suppose that (2.11) is false. Therefore, there exists a sequence $t_n \to \infty$ such that $u(2^{-1} t_n)$ is bounded.

We again consider the approximating problems (2.6), (2.7) and using (2.9) and the following
estimate (cf. (2.8))
\[ \left| \frac{d^2 v}{dt^2} (t) \right| \leq |A^0 x| + t^{-1} |x - u(t)| \leq 0, \quad t > 0, \quad x \in D(A) \]

one obtains that \( F = A^{-1} 0 \) is nonempty. Indeed, the arguments above and the assumption that \( 0 \in \mathcal{R}(\bar{A}) \) implies that \( \{t_n\} \) has a subsequence \( \{t_{n_k}\} \) such that
\[
y_{n_k} (t_{n_k}) \text{ converges weakly to an element } p \in H, \quad \frac{d^2 v_n}{dt^2} (t_{n_k}) \text{ converges strongly to 0, as } k \to \infty.
\]

Using (2.6) one obtains \( \{ p, 0 \} \in A \), i.e. \( F \neq \emptyset \), which is a contradiction.

Next, we shall investigate the case \( A \) is a square root. Let \( B \) be a maximal monotone set in \( H \times H \) such that \( B^{-1} 0 \) is nonempty. Consider the following second order problem on half axis:
\[
\begin{align*}
\frac{d^2 v}{dt^2} (t) & \in B v(t), \quad t > 0 \quad (2.12) \\
v(0) & = x, \quad x \in D(B) \quad (2.13) \\
sup_{t \geq 0} |v(t)| & < \infty. \quad (2.14)
\end{align*}
\]

It is proved in [1] that this problem has a unique solution \( v \) satisfying
\[
v \in W^{2, 2} (0, \infty; H) \quad \text{and} \quad \frac{d v}{dt} \in L^2 (0, \infty; H).
\]

Let \( A = B^{1/2} \) be the square root of \( B \) in the sense of definition given by Barbu [1, p. 329]. Then, \( v \) satisfies the following equation
\[
\frac{dv}{dt} (t) + A^0 v(t) = 0, \quad \forall t > 0. \quad (2.15)
\]

**Lemma 2.2.** \( A^{-1} 0 = B^{-1} 0 \), where \( A = B^{1/2} \).

**Proof.** First, let us take \( p \in B^{-1} 0 \). It follows that \( v(t) = p, t \geq 0 \) is a solution of (2.12), (2.13), (2.14), where \( x = p \). But, by the definition of the square root \( v \) is at the same time a solution of (2.15), that is,
\[
A^0 p = 0.
\]

Hence
\[
B^{-1} 0 \subset A^{-1} 0.
\]

Similarly, it follows
\[
A^{-1} 0 \subset B^{-1} 0.
\]

**Theorem 2.5.** Let \( B \) be a maximal monotone set in \( H \times H \) and suppose that \( B^{-1} 0 \neq \emptyset \). Let \( u \)

be the solution of (1.1), (1.2), where \( A = B^{1/2} \) and (1.4), (1.5) hold. Then, \( u(t) \) converges weakly, as \( t \to \infty \), to an element \( p \in B^{-1} 0 \).

**Proof.** We recall that
\[
D(B) \subset D(A) \subset D(\bar{A}),
\]

therefore
\[
D(A) = D(\bar{B}).
\]

We also recall that the solution of (2.15), (2.13) satisfies
\[
\left| \frac{dv}{dt} (t) \right| \leq t^{-1} |x - y|, \quad \text{for } t > 0, \quad y \in B^{-1} 0.
\]

By Lemma 2.2 \( A^{-1} 0 \) is nonempty and it coincides to \( B^{-1} 0 \). Consequently, using the same procedure as in the case \( A = \partial \varphi \) the theorem follows.

**Remark 2.2.** Making \( f = 0 \) in Theorem 2.5, we obtain the weak convergence (to a point of \( B^{-1} 0 \)) of the solution of second order problem (2.12), (2.13), (2.14).

**Theorem 2.6.** Suppose that \( A = \partial \varphi, A^{-1} 0 \neq \emptyset \) and (1.4), (1.6) hold, therefore (1.1), (1.2) has a strong solution \( u \in C(0, \infty; H) \cap W_{loc}^{1, 2} (\delta \cup \infty; H), \forall \delta \in ]0, \infty[ \). Then,
\[
\lim_{t \to a} \varphi (u(t)) = \varphi (a), \quad a \in ]0, \infty[ \quad (2.16)
\]

**Proof.** Multiplying (1.1) by \( du/dt (t) \) and using Lemma 2.1 in [1, p. 189] we obtain:
\[
\left| \frac{du}{dt} (t) \right|^2 + \frac{d}{dt} \varphi (u(t)) = \left( f(t), \frac{du}{dt} (t) \right), \quad \forall t > 0.
\]

Hence
\[
\left| \frac{du}{dt} (t) \right|^2 + \frac{d}{dt} \varphi (u(t)) \leq \frac{1}{2} |f(t)|^2, \quad \forall t > 0 \quad (2.18)
\]

which implies that the function
\[
t \to \varphi (u(t)) - 2^{-1} \int_{0}^{t} |f(s)|^2 ds
\]

is monotone nonincreasing over \( ]0, \infty[ \). Moreover, since \( A^{-1} 0 \neq \emptyset \), we have
\[
\varphi (t) \geq \varphi (p), \quad p \in A^{-1} 0, \quad \forall t \in H.
\]

Thus, (2.19) has a finite limit, as \( t \to \infty \), thereby (2.17) is satisfied. By (1.6) and (2.18) one gets (2.16).

**Remark 2.3.** Under the hypotheses of Theorem 2.6 \( u(t) \) can be unbounded, as \( t \to \infty \). For instance,
if we take:

\[ H = (-\infty, \infty), \quad A \equiv 0 \quad \text{and} \quad f(t) = (1 + t)^{-1}, \]

then the assertion follows.

In addition to the hypotheses of Theorem 2.6 let us assume that \( u(t) \) is bounded on \([0, \infty[\). For example, this is true if (1.5) holds (cf. Lemma 2.1) or if

\[
\lim_{t \to \infty} \varphi(u) = \infty.
\]

(2.20)

In this case, since

\[
\varphi(u(t)) \leq \varphi(v) + (h(t), u(t) - v), \quad \text{for every} \quad v \in D(\varphi)
\]

and

\[
h(t) \in L^2(\delta, \infty; H), \quad h(t) \in \partial \varphi(u(t)), \quad \text{a.e.} \quad t > 0,
\]

it follows that

\[
\varphi_u = \varphi(u_u) \leq \varphi(v), \quad \forall v \in D(\varphi), \quad u_u \in \omega_u.
\]

Hence, in this situation we have:

\[
\varphi_u = \lim_{t \to \infty} \varphi(u(t)) = \inf \{ \varphi(u); u \in H \}.
\]

Next, we shall apply our preceding results to some second order differential equations. Suppose that we are given two real Hilbert spaces \( V \) and \( H \) such that \( V \subset H \) and the inclusion mapping of \( V \) into \( H \) is continuous and densely defined. We are interested in the problem:

\[
\frac{d^2 u}{dt^2}(t) + A(t)u(t) + M(t) \frac{du}{dt}(t) \geq f(t), \quad \text{a.e.} \quad t > 0
\]

(2.21)

\[
u(0) = u_0, \quad \frac{du}{dt}(0) = v_0.
\]

(2.22)

where

\[ A(t) : V \to V' \]

is linear, continuous, symmetric and coercive, while \( M(t) \) is a maximal monotone set in \( V \times V', V' \) being the dual space of \( V \). Let \( X = V \times H \) and let \( K \subset X \times X \) be defined by

\[
D(K) = \{ [u, v] \in X; (A(t)u + M(t)v) \cap H \neq \emptyset \},
\]

\[ K[u, v] = [-v, (A(t)u + M(t)v) \cap H], \quad \text{for every} \quad [u, v] \in D(K). \]

Then \( K \) is a maximal monotone set relating to an adequate scalar product of \( X \) (see e.g. [1, p. 268]).

It is well known that if \( [u_0, v_0] \in D(K) \) and \( f \in L^1_{loc}(0, \infty; H) \) then (2.21), (2.22) has a unique strong solution \( u \in W_{loc}^{1,1}(0, \infty; V) \cap W_2^{1,2}(0, \infty; H) \), or equivalently the following Cauchy problem:

\[
\frac{d^2 U}{dt^2}(t) + KU(t) \geq F(t), \quad \text{a.e.} \quad t > 0,
\]

(2.23)

\[ U(0) = [u_0, v_0].
\]

(2.24)

where \( F(t) = [0, f(t)] \), admits \( U(t) = [u(t), du/dt(t)] \) as solution. It is easy to see that if we suppose \([u_0, v_0] \in D(K)\) and \( f \in L^1_{loc}(0, \infty; H) \) then there exists a unique function

\[ u \in C([0, \infty[, V) \cap C^1([0, \infty[, H) \]

such that \([u, du/dt] \) is an integral solution of (2.23), (2.24) on positive half-axis. We shall say that this function \( u \) is an integral solution of (2.21), (2.22).

**Corollary 2.1.** Suppose that \( A_0, M \) satisfy the precedent assumptions, \( [u_0, v_0] \in D(K)\), \( f \in L^1(0, \infty; H) \) and \( 0 \in D(M) \). Then

\[
u(t) \text{ is bounded in } V \text{ on } [0, \infty[, \]

(2.25)

\[
deu \text{ is bounded in } H \text{ on } [0, \infty[, \]

(2.26)

\[ \sigma(t) \text{ converges weakly in } V \]

(2.27)

\[ \text{to an element } p \in C = -A_0^{-1} M(0), \text{ as } t \to \infty, \]

(2.28)

where

\[ \sigma(t) = t^{-1} \int_0^t u(s) \, ds. \]

(2.29)

3. **Examples**

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \), \( \mathbb{R}^1 = (-\infty, +\infty) \) and \( \Gamma \) denotes its boundary which is smooth enough. Let \( \beta \) be a maximal monotone graph in \( \mathbb{R}^1 \times \mathbb{R}^1 \) such that \( 0 \in D(\beta) \). Then, there exists a lower-semicontinuous convex function \( f : \mathbb{R}^1 \to [-\infty, \infty] \) such that \( \beta \circ \partial f \).

**Example 1.**

\[ u(t, x) - \Delta u(t, x) + \beta(u(t, x)) \geq f(t, x), \quad t > 0, \quad \text{a.e.} \quad x \in \Omega \]

(3.1)

\[ u(t, x) = 0, \quad x \in \Gamma, \quad t > 0 \]

(3.2)

\[ u(0, x) = u_0(x), \quad \text{a.e.} \quad x \in \Omega \]

(3.3)
The function \( \varphi: H = L^2(\Omega) \to ]-\infty, \infty[ \).

\[
\varphi(u) = \begin{cases} 
2^{-1} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega f(u) \, dx, & u \in H^1_0(\Omega) \\
+ \infty & \text{otherwise}, 
\end{cases}
\]

is lower-semicontinuous, convex and \( \varphi \neq +\infty \),

\[
\partial \varphi(u) = \{ v \in L^2(\Omega) : (x) \in f(u(x)) - \Delta u(x), \text{ a.e. } x \in \Omega \}.
\]

Assuming that

\[
u_0 \in D(\varphi)' = \{ u \in L^2(\Omega) : u(x) \in D(\varphi)' \}, \text{ a.e. } x \in \Omega \tag{3.4}
\]

\[
f(t, x) \in L^2(0, \infty ; L^2(\Omega)) \tag{3.5}
\]

and applying Theorem 2.3, we deduce that \( u(t, x) \), the integral solution of (3.1), (3.2), (3.3), converges weakly in \( L^2(\Omega) \), as \( t \to \infty \), to \( u_\infty(x) \in H^1_0(\Omega) \cap H^2(\Omega) \), where \( u_\infty(x) \) is the unique solution of the problem:

\[
\begin{aligned}
- \Delta u_\infty(x) + f(u_\infty(x)) &\geq 0, \text{ a.e. } x \in \Omega \\
u_\infty(x) &= 0, & x \in \Gamma.
\end{aligned}
\]

Suppose now that (3.4) holds and

\[
f(t, x) \in L^2(0, \infty ; L^2(\Omega)). \tag{3.6}
\]

From Theorem (2.6) and Remark 2.3 we can write that

\[
\lim_{t \to \infty} \varphi(u(t, \cdot)) = \inf \{ \varphi(u) : u \in L^2(\Omega) \}. \tag{3.7}
\]

By (3.7), taking into account that \( j \) is bounded from below by an affine function, it follows that \( u(t, \cdot), t > 0 \) is bounded in \( H^1_0(\Omega) \), so it is relatively compact in \( L^2(\Omega) \). But (3.7) implies \( u_\infty = \{ u_\infty \} \). We conclude that if (3.4), (3.6) are satisfied then \( u(t, \cdot) \) converges strongly in \( L^2(\Omega) \) and weakly in \( H^1_0(\Omega) \) to \( u_\infty \), as \( t \to \infty \).

**Example 2.**

\[
u_0(t, x) = \Delta u(t, x) + f(u(t, x)) \geq f(t, x), t > 0, \text{ a.e. } x \in \Omega \tag{3.8}
\]

\[
u(t, x) = 0, \quad u(0, x) = u_0(x), t \geq 0 \tag{3.9}
\]

\[
u(0, x) = u_0(x), \quad u(0, x) = v_0(x), \text{ a.e. } x \in \Omega. \tag{3.10}
\]

According to Corollary 2.1, we shall choose

\[
V = H^1_0(\Omega), \quad H = L^2(\Omega), \quad A_0 = -\Delta, \quad \text{i.e.}
\]

\[
(\omega, u)_H = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \text{for all } [u, v] \in H^1_0(\Omega)^2
\]

and \( M = \partial \varphi \), where

\[
\omega(u) = \int_\Omega f(u) \, dx, \quad u \in H^1_0(\Omega).
\]

We remark that \( 0 \in D(\varphi) \), because \( 0 \in D(f) \). Hence, assuming that (3.5) is satisfied it follows (2.25)-(2.28).

Moreover,

\[
\sigma(t, \cdot) = t^{-1} \int_0^t u(s, \cdot) \, ds
\]

converges strongly in \( L^2(\Omega) \) to \( u_\infty \in \Delta^{-1} \mathcal{R}(0) \).

For details concerning these examples, see e.g. [1].

**REFERENCES**