ON A PROBLEM ARISING IN CAPILLARITY THEORY

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1 Introduction

In this paper we consider a class of nonlinear equations which naturally arise in capillarity theory. If we denote by $u = u(x, y)$ the height of a liquid in a vertical tube above the reference plane $u = 0$ then the equilibrium shape of the liquid surface with constant surface tension in a uniform gravity field is described by the classical equation of capillarity [7, pp. 262-263], [8, pp. 289-293]

\[
\text{div} \frac{\nabla u}{(1 + ||\nabla u||^2)^{1/2}} = ku, \quad (x, y) \in \Omega ,
\]

where $\Omega$ is the domain of the plane $u = 0$ occupied by the tube and $k$ is a positive constant. The natural physical boundary condition is

\[
(1 + ||\nabla u||^2)^{-1/2} \frac{\partial u}{\partial n} = \cos \gamma ,
\]

where $\gamma$ is the contact angle (i.e., the angle between the liquid surface and the boundary $\partial \Omega$, measured within the fluid) and $n$ is the corresponding outward normal to $\partial \Omega$.

Now, if we consider the functional (see [2] for $\Omega = \mathbb{R}^n$)

\[
J(v) = \int_{\Omega} \left\{ \frac{1}{2} \Phi(||\nabla v||^2) + h(v) \right\} dx - \int_{\partial \Omega} \phi v dx,
\]

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then formally every critical point \( u \) of \( J \) (i.e., \( J'(u) = 0 \)) is a solution of the following boundary value problem

\[
\text{(1.4)} \quad \text{div}(\Phi'(||\nabla u||^2)\nabla u) = h'(u), \quad \text{in } \Omega,
\]

\[
\text{(1.5)} \quad \Phi'(||\nabla u||^2) \frac{\partial u}{\partial n} = \phi, \quad \text{on } \partial \Omega.
\]

Obviously, if we take

\[
\Phi(\xi) = 2((\xi + 1)^{1/2} - 1), \quad \phi = \cos \gamma, \quad h(\xi) = k\xi^2/2,
\]

then problem (1.4), (1.5) coincides with capillarity problem (1.1), (1.2). In this paper we shall study the particular case in which \( \Omega \) is the disk \( D(0,1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \), and \( u \) depends only on \( r = (x^2 + y^2)^{1/2} \) (so that it is natural to assume that \( \phi \equiv \phi_0 = \text{Const.} \)). In the case of capillarity problem this means that the tube is circular and the height of the liquid depends only on the distance \( r \) from the origin. From a physical point of view this last assumption is very natural because the liquid surface in circular tubes is a rotation surface. So we can pass in (1.4), (1.5) to polar coordinates

\[
(x = r\cos \theta, \quad y = r\sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi)
\]

and after standard computations we get

\[
\text{(1.6)} \quad (ru'(r)\Phi'(u'(r)^2))' = rh'(u(r)), \quad 0 < r < 1,
\]

\[
\text{(1.7)} \quad u'(1)\Phi'(u'(1)^2) = \phi_0.
\]

The corresponding functional \( J \) takes the form

\[
\text{(1.8)} \quad J(u) = 2\pi \left\{ \int_0^1 r[(1/2)\Phi(v'(r)^2) + h(v(r))] \, dr - \phi_0 v(1) \right\}.
\]

Denoting

\[
\psi(\xi) := (1/2)\Phi(\xi^2), \quad G(\xi) := j'(\xi) = \xi \Phi'(\xi^2), \quad \Psi := (2\pi)^{-1} J \quad \text{and} \quad H := h'
\]

we can write (1.6), (1.7), (1.8) as follows

\[
\text{(1.9)} \quad (rG(u'(r)))' = rH(u(r)), \quad 0 < r < 1,
\]

\[
\text{(1.10)} \quad G(u'(1)) = \phi_0.
\]
(1.11) \[ \Psi(v) = \int_0^1 r[j(v'(r)) + h(v(r))] \, dr - \phi_0 v(1) . \]

We remark that the existence of solution to (1.4), (1.5) is not a trivial problem. To have an idea about this question let us observe that in the case of capillarity model the functional \( J \) defined by (1.3) is coercive at most on the space \( W^{1,1}(\Omega) \) (which is not reflexive). However in the one-dimensional case (1.9), (1.10) we are able to prove existence under very reasonable assumptions. To this purpose we can use either a variational or a direct approach. In that which follows we shall illustrate both these approaches.

2 The variational approach

Throughout this section we shall assume the following hypotheses:

(H.1) \( G : R \to R \) is continuous, strictly increasing, and \( G(0) = 0 ; \)

(H.2) \( H : R \to R \) is continuous, strictly increasing, \( H(0) = 0 \), and there exist constants \( C_1, C_2 > 0 \) such that

(2.1) \[ H(r) \geq C_1 r - C_2 , \ r \in R . \]

Taking into account (H.1) we can see that (1.10) may be expressed as

(2.2) \[ u'(1) = \beta . \]

Obviously in the case of capillarity model assumptions (H.1), (H.2) are satisfied.

The main result of this section is

**Theorem 2.1** If assumptions (H.1), (H.2) are satisfied then problem (1.9), (2.2) has a unique solution \( u \in C^1[0,1] \).

We first prove uniqueness:

**Lemma 2.1** Problem (1.9), (2.2) has at most one solution \( u \in C^1[0,1] \).

**Proof.** If \( u_1, u_2 \in C^1[0,1] \) are solutions for (1.9), (2.2) then

\[ \int_0^1 (u_1 - u_2) [r[G(u'_1) - G(u'_2)]'] \, dr = \]

\[ = \int_0^1 r(u_1 - u_2) [H(u_1) - H(u_2)] \, dr . \]

Integrating by parts in the left hand side and using the strict monotonicity of \( G, H \) and the fact that \( u'_1(1) = u'_2(1) = \beta \) we get \( u_1 = u_2 \). Q.E.D.

We continue with the following auxiliary result
Lemma 2.2 If \( u \in C^1[0,1] \) is a solution of eq. (1.9) then the following implications hold:

\[
(2.3) \quad u(0) = 0 \Rightarrow \text{either } u' \geq 0 \text{ in } [0,1] \text{ or } u' \leq 0 \text{ in } [0,1];
\]

\[
(2.4) \quad u(0) > 0 \Rightarrow u' > 0 \text{ in } (0,1);
\]

\[
(2.5) \quad u(0) < 0 \Rightarrow u' < 0 \text{ in } (0,1).
\]

Proof. We multiply eq. (1.9) by \( u(r) \) and then integrate on \([0,r]\):

\[
(2.6) \quad r u'(r) G(u'(r)) = \int_0^r s[u'G(u') + uH(u)] \, ds, \quad 0 \leq r \leq 1.
\]

Using (H.1), (H.2) we can see that

\[
\{ r \in (0,1) : u(r) = 0 \} = \{ r \in (0,1) : u'(r) = 0 \}
\]

and this set is either an empty set or an interval of the form \((0, \delta]\). Using this remark and eq. (2.6) we can easily derive the conclusions (2.3), (2.4), and (2.5). Q.E.D.

To continue the proof of Theorem 2.1 we shall consider that \( \beta > 0 \) (the case \( \beta < 0 \) may be similarly solved and for \( \beta = 0 \) problem (1.9), (2.2) has null solution).

As we shall see later, it is convenient to replace \( G \) by \( \tilde{G} : R \to R \) defined by

\[
\tilde{G}(\xi) = \begin{cases} 
\xi & \text{for } \xi < 0, \\
G(\xi) & \text{for } 0 \leq \xi \leq \beta, \\
\xi - \beta + G(\beta) & \text{for } \xi > \beta.
\end{cases}
\]

In a first stage we shall consider the approximate equation

\[
(2.7) \quad (r \tilde{G}(u'))' = r H_\lambda(u), \quad 0 \leq r \leq 1
\]

where \( H_\lambda, \lambda > 0 \) is the Yosida approximation of \( H \) (see, e.g., [9, p. 20]):

\[
H_\lambda = \lambda^{-1}(I - J_\lambda), \quad J_\lambda = (I + \lambda H)^{-1}.
\]

We recall that \( H_\lambda = h_\lambda', \lambda > 0 \), where \( h_\lambda \) is the Moreau-Yosida approximation of \( h(\xi) = \int_0^\xi H(s) \, ds \) (see, e.g., [9, p. 39]). We have
Lemma 2.3 For $\lambda_0 > 0$ fixed there exist constants $C_3, C_4 > 0$ such that

$$h_\lambda(\xi) \geq C_3 \xi^2 - C_4, \quad \xi \in R, \quad 0 < \lambda \leq \lambda_0.$$  

Proof. By (2.1) we have

$$h(\xi) \geq C_5 \xi^2 - C_6, \quad \xi \in R,$$

where $C_5, C_6$ are positive constants. So

$$h_\lambda(\xi) \geq \inf\{(\xi - \theta)^2/2\lambda + C_5 \theta^2; \theta \in R\} - C_6$$

$$\geq C_5 \xi^2/(1 + 2\lambda C_5) - C_6 \geq C_5 \xi^2(1 + 2\lambda_0 C_5) - C_6,$$

for $0 < \lambda \leq \lambda_0, \xi \in R$. Q.E.D.

Lemma 2.4 For each $\lambda \in (0, \lambda_0]$, problem (2.7), (2.2) has a unique solution $u_\lambda \in C^1[0, 1]$.

Proof. Consider the space

$$X := \{v = v(r); \quad r^{1/2} v, \quad r^{1/2} v' \in L^2(0, 1)\}.$$ 

This is a real Hilbert space with scalar product

$$< v_1, v_2 >_X = \int_0^1 r(v_1 v_2 + v_1' v_2') \, dr$$

and with the corresponding norm. Consider the functional $\Psi^\lambda: X \rightarrow R, \quad 0 < \lambda \leq \lambda_0$, associated to problem (2.7), (2.2):

$$\Psi^\lambda(v) = \int_0^1 r[\tilde{j}(v') + h_\lambda(v)] \, dr - G(\beta)v(1),$$

where $\tilde{j}(\xi) = \int_0^\xi \tilde{G}(s) \, ds$. From the definition of $\tilde{G}$ we easily obtain

$$\xi^2/3 - C \leq \tilde{j}(\xi) \leq \xi^2 + C, \quad \xi \in R,$$

where $C$ is a positive constant. Since

$$0 = h_\lambda(0) \leq h_\lambda(\xi) \leq \xi H_\lambda(\xi) \leq \xi^2/\lambda, \quad \xi \in R, \quad \lambda > 0$$

it follows that $D(\Psi^\lambda) = X, \quad \lambda > 0$. On the other hand, we have [10, p. 3]

$$|v(1)| \leq C_7 ||v||_X, \quad v \in X.$$
Using (2.8), (2.10) and (2.11) we can see that
\[ \Psi^\lambda(v) \geq C_8 \|v\|^2_X - C_0, \quad v \in X, \quad 0 < \lambda \leq \lambda_0, \]
where $C_8$, $C_0$ are some positive constants. In addition $\Psi^\lambda$ is convex and continuous on $X$. Hence for each $\lambda \in (0, \lambda_0)$, $\Psi^\lambda$ has a minimum point $u_\lambda \in X$ [9, Thm. 1.10, p. 34]. Therefore we have (Euler-Lagrange eq.)
\[ (r\ddot{G}(u_\lambda'))' = rH_\lambda(u_\lambda), \quad 0 < \lambda \leq \lambda_0, \quad 0 < r \leq 1. \]  
(2.12)

Hence $u_\lambda \in C^1(0, 1)$ and $\ddot{G}(u_\lambda(1)) = \ddot{G}(\beta) = G(\beta)$ which implies
\[ u_\lambda'(1) = \beta. \]  
(2.13)

By (2.12) we get
\[ r\ddot{G}(u_\lambda(r)) = G(\beta) - \int_r^1 sH_\lambda(u_\lambda(s))ds, \quad 0 < r \leq 1. \]  
(2.14)

As $u_\lambda \in X$ and $|H_\lambda(u_\lambda(s))| \leq |u_\lambda(s)|/\lambda$ it follows by (2.14) that there exists
\[ \lim_{r \to 0^+} r\ddot{G}(u_\lambda'(r)) = l_\lambda \in R. \]  
(2.15)

Using (2.15) and the definition of $\ddot{G}$ we get
\[ \lim_{r \to 0^+} ru_\lambda'(r) = l_\lambda. \]

Actually $l_\lambda = 0$ because otherwise
\[ r(u_\lambda'(r))^2 \geq l_\lambda^2/(2r) \]
on some interval $(0, \delta]$ which contradicts the fact that $u_\lambda \in X$. Now, using (2.12) and (2.15) (with $l_\lambda = 0$) we get
\[ r\ddot{G}(u_\lambda'(r)) = \int_0^r sH_\lambda(u_\lambda)ds, \quad 0 \leq r \leq 1. \]  
(2.16)

Since $|H_\lambda(\xi)| \leq \xi/\lambda$ for $\lambda > 0$, $\xi \in R$, we have
\[ |\int_0^r sH_\lambda(u_\lambda)ds| \leq \lambda^{-1} \int_0^r s|u_\lambda(s)|ds \leq \]
\[ \leq (r/\lambda\sqrt{2})(\int_0^r s|u_\lambda(s)|^2ds)^{1/2} \]
and hence \( \tilde{G}(u'_\lambda(r)) \to 0 \) for \( r \to 0^+ \) \( \Rightarrow u'_\lambda(r) \to 0 \) for \( r \to 0^+ \). Therefore \( u_\lambda \in \mathcal{C}^1[0,1] \) and \( u'_\lambda(0) = 0 \). The uniqueness of \( u_\lambda \) follows by Lemma 2.1 (which is also valid for eq. (2.7), because \( H_\lambda = HJ_\lambda \) is strictly increasing).

Q.E.D.

The next step of the proof of Theorem 2.1 is

**Lemma 2.5** For every \( \lambda \in (0, \lambda_0] \), \( u'_\lambda \) is nondecreasing, where \( u_\lambda \) is the solution of (2.7), (2.2).

**Proof.** Since \( \tilde{G}, H_\lambda \) satisfy (H.1) and (H.2) we can see that Lemma 2.2 is valid for eq. (2.7). Therefore, as \( u'_\lambda(1) = \beta > 0 \), we have

\[
(2.17) \quad u'_\lambda \geq 0, \quad u_\lambda \geq 0 \text{ in } [0,1].
\]

As \( H_\lambda \) is Lipschitzian and \( H_\lambda(0) = 0 \) it follows that (see also (2.17)) \( H_\lambda(u_\lambda(\cdot)) \) is Lipschitzian, nondecreasing, and \( \geq 0 \) in \( [0,1] \). So it is easily seen that the function

\[
w_\lambda(r) = (1/r) \int_0^r sH_\lambda(u_\lambda(s)) \, ds
\]

is nondecreasing in \([0,1]\). Therefore \( u'_\lambda \) is also nondecreasing in \([0,1]\) because (see (2.16))

\[
(2.18) \quad u'_\lambda(r) = \tilde{G}^{-1}(w_\lambda(r)), \quad 0 \leq r \leq 1.
\]

Q.E.D.

**Proof of Theorem 2.1 (continuation).** By Lemma 2.5 we have for \( 0 < \lambda \leq \lambda_0 \) and \( 0 \leq r \leq 1 \)

\[
(2.19) \quad 0 = u'_\lambda(0) \leq u'_\lambda(r) \leq u'_\lambda(1) = \beta
\]

and hence \( u_\lambda \) is the (unique) solution of the following equation

\[
(2.20) \quad rG(u'_\lambda(r)) = \int_0^r sH_\lambda(u_\lambda(s)) \, ds, \quad 0 \leq r \leq 1, \quad 0 < \lambda \leq \lambda_0
\]

satisfying (2.2) : \( u'_\lambda(1) = \beta \). Taking into account the fact that \( u_\lambda \) is a minimum point of \( \Psi_\lambda^\lambda \) we have

\[
\Psi_\lambda^\lambda(u_\lambda) \leq \Psi_\lambda^\lambda(0) = 0, \quad 0 < \lambda \leq \lambda_0.
\]

Therefore, according to (2.10) and Lemma 2.3, we deduce that the set

\[
(2.21) \quad \{u_\lambda; 0 < \lambda \leq \lambda_0\} \text{ is bounded in } X.
\]
By Arzelà's Criterion (2.21) implies that \( \{u_\lambda; 0 < \lambda \leq \lambda_0\} \) is relatively compact in \( C[\delta, 1] \) for some \( \delta \in (0, 1) \). In fact, taking into account (2.19), it follows that the set \( \{u_\lambda; 0 < \lambda \leq \lambda_0\} \) is bounded in \( W^{1, \infty}(0, 1) \). This implies that there exists \( u \in W^{1, \infty}(0, 1) \) such that

\[
(2.22) \quad u_\lambda \to u \text{ in } C[0, 1],
\]

\[
(2.23) \quad u'_\lambda \rightharpoonup u' \text{ weakly-star in } L^\infty(0, 1)
\]

for \( \lambda \to 0^+ \) (on a subsequence of \( \lambda \) again denoted by \( \lambda \)). In addition, for \( \lambda \to 0^+ \), \( J_\lambda u_\lambda(s) \to u(s) \) and hence

\[
(2.24) \quad H_\lambda(u_\lambda(s)) = H(J_\lambda u_\lambda(s)) \to H(u(s)), \quad 0 \leq s \leq 1.
\]

Using (2.24) and the fact that

\[
|J_\lambda u_\lambda(s)| \leq |u_\lambda(s)| \leq \text{Const.}, \quad 0 \leq s \leq 1, \quad 0 < \lambda \leq \lambda_0
\]

we can use Lebesgue's Dominated Convergence Theorem to pass to limit in the right hand side of (2.20). Now, using (2.19), (2.23) and (2.20) we can see that \( u \in C^1[0, 1], 0 \leq u' \leq \beta \) and \( u \) satisfies (1.9) and (2.2). Q.E.D.

Remark 2.1. It is evident from the proof of Theorem 2.1 that in (H.1), (H.2) we can take \( D(G) = [0, \beta] \), \( D(H) = R_+ \) for \( \beta > 0 \) (and, similarly, \( D(G) = [\beta, 0] \), \( D(H) = R_- \) for \( \beta < 0 \)).

Remark 2.2. Following [10] we can derive additional properties of solutions (for example continuous dependence of solutions on \( \beta \)).

Remark 2.3. Using the same method we can study the following more general degenerate equation

\[
(2.25) \quad (p(r)G(u'))' = q(r)H(u), \quad 0 \leq r \leq 1
\]

where \( p, q \geq 0 \), \( p(0) = q(0) = 0 \). Moreover, we believe that eq. (2.25) with condition (2.2) (or even with a more general condition) can be solved in a general Hilbert space, as V. Barbu [1, Chap. V] did for a similar non-degenerate equation.

On the other hand, eq. (1.4) with \( \Phi(\xi) = \xi \) and \( h(\xi) = k\xi^2/2 \) represents a simplified model for capillarity, as considered by Landau and Lifschitz [8]. The corresponding one-dimensional eq.(1.9) coincides in this case with the well-known modified Bessel equation of order 0 (see, e.g., [3, pp.92-93]).
3 A direct approach

To illustrate this approach we consider the following equation

\[ r^a G(u') = \int_0^r s^b u(s) \, ds , \quad 0 \leq r \leq 1 \]

with boundary condition (2.2), where \( G \) satisfies (H.1) and
\[ a, b \in \mathbb{R}, \quad b + 1 > \max(0, a). \]

**Remark 3.1.** If assumptions (H.1), (H.3) hold and in addition \( a > 0 \) then the problem of existence of solutions \( u \in C^1[0, 1] \) for (3.1), (2.2) is equivalent with the same problem for the following equation

\[ (r^a G(u'))' = r^b u , \quad 0 \leq r \leq 1 \]

with condition (2.2). Clearly, for \( a = b = 1 \) and \( G(\xi) = \xi/(1 + \xi^2)^{1/2} \), eq.(3.2) is exactly the one-dimensional equation of capillarity.

The main result of this section is

**Theorem 3.1** If assumptions (H.1) and (H.3) hold then problem (3.1), (2.2) has a unique solution \( u \in C^1[0, 1] \).

**Sketch of proof** (for details see [6]). If \( u \in C^1[0, 1] \) is a solution of eq.(3.1) then obviously

\[ r^a G(u'(r)) \to 0 \quad \text{as} \quad r \to 0^+. \]

Using (3.3) we can show uniqueness of solution to problem (3.1), (2.2) by a reasoning similar to that used in the proof of Lemma 2.1. To prove existence for (3.1), (2.2) we associate to eq. (3.1) the initial condition

\[ u(0) = u_0. \]

Denoting \( F := G^{-1} \) and \( y := u' \) we can see that problem (3.1), (3.4) is equivalent with the following equation

\[ y(r) = F(u_0 r^{b+1-a} / (b+1) + r^{-a} \int_0^r s^b \int_0^s y(t) \, dt \, ds). \]

In a first stage we assume, in addition to (H.1), that

\[ G(\pm \infty) = \pm \infty \quad (\text{hence} \quad F(\pm \infty) = \pm \infty) \quad \text{and} \quad F \text{ is Lipschitzian}. \]

By a simple reasoning involving Banach's Fixed Point Principle we can prove the following result:
Lemma 3.1 If (H.1), (H.3) and (3.6) are satisfied then for every \( u_0 \in R \) eq. (3.5) has a unique solution \( y = y(\tau, u_0) \in C[0,1] \).

To continue the proof of Theorem 3.1 we shall assume as before that \( \beta > 0 \). By using some elementary arguments we can show the following properties of \( y(\tau, u_0) \) (under assumptions (H.1), (H.3) and (3.6)):

\[
(3.7) \quad y(0, u_0) = 0, \quad \forall u_0 \in R;
\]

\[
(3.8) \quad y(\tau, 0) = 0 \quad \text{for} \quad 0 \leq \tau \leq 1;
\]

\[
(3.9) \quad y(\tau, u_0) > 0 \quad \text{for} \quad 0 < \tau \leq 1, \quad u_0 > 0;
\]

\[
(3.10) \quad 0 < u_0 < \bar{u}_0 \Rightarrow y(\tau, u_0) < y(\tau, \bar{u}_0) \quad \text{for} \quad 0 < \tau \leq 1;
\]

\[
(3.11) \quad \forall u_0 > 0, \text{ the function } \tau \mapsto y(\tau, u_0) \text{ is strictly increasing in } (0,1);
\]

\[
(3.12) \quad \text{There exists } C > 0 : |y(\tau, u_0) - y(\tau, \bar{u}_0)| \leq C|u_0 - \bar{u}_0|, \forall u_0, \bar{u}_0 > 0, \quad 0 \leq \tau \leq 1.
\]

By (3.8), (3.13) and (3.12) we obtain (cf. Darboux's property) that

\[
\{y(1, u_0) : u_0 > 0\} = (0, \infty).
\]

Therefore problem (3.1), (2.2) has a solution \( u \in C^1[0,1], \ u(\tau) = u(\tau, u_0) = u_0 + \int_0^\tau y(s, u_0) \, ds \), for some \( u_0 > 0 \). According to (3.10), \( u_0 \) is unique and so \( u \) is unique too.

The next step of the proof is to drop the assumption that \( F \) is Lipschitzian.

To this purpose we replace \( F \) by its Yosida approximation \( F_\lambda, \ \lambda > 0 \):

\[
F_\lambda = (1/\lambda)(I - J_\lambda) = FJ_\lambda, \quad J_\lambda = (I + \lambda F)^{-1}.
\]

According to the above reasoning, for every \( \lambda > 0 \) there is \( u_\lambda \in C^1[0,1] \) satisfying:

\[
(3.14) \quad u_\lambda'(r) = F_\lambda(r^{-a} \int_0^r s^b u_\lambda(s) \, ds), \quad 0 \leq r \leq 1.
\]
(3.15) \[ u_\lambda'(1) = \beta , \]

(3.16) \[ 0 \leq u_\lambda'(r) \leq \beta , \quad 0 \leq r \leq 1 . \]

We can show that \( u_\lambda \) converges in \( C[0, 1] \) as \( \lambda \to 0^+ \), on a subsequence, to some function \( u \). In fact, \( u \in C^1[0, 1] , \quad 0 \leq u' \leq \beta \) and \( u \) satisfies (3.1), (2.2).

To show that we pass to limit as \( \lambda \to 0^+ \) in (3.14), (3.15).

The final step is to drop the assumption \( G(\pm \infty) = \pm \infty \). To do that we replace \( G \) by a function \( \tilde{G} \) defined as in Section 2. Obviously, \( \tilde{G} \) satisfies (H.1) and, in addition, \( \tilde{G}(\pm \infty) = \pm \infty \). But, as seen before, the corresponding solution \( u \) for (3.1), (2.2) where \( G \) is replaced by \( \tilde{G} \) satisfies \( 0 \leq u' \leq \beta \) and so \( u \) is in fact a solution of (3.1), (2.2).

Q.E.D.

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References


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