On a Second Order Boundary Value Problem Related to Capillary Surfaces

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Abstract. The existence and uniqueness of the solution for the nonlinear (possibly degenerate) second order BVP (1.1), (1.2) below are investigated under certain assumptions on $p, q, G$ and $H$. An application in the theory of capillarity and some generalizations are also discussed.

1. Introduction

In this paper we investigate the existence and uniqueness of the solution of the following nonlinear (possibly degenerate) second order boundary value problem (BVP)

$$
(p(r)G(u'(r)))' = q(r)H(u(r)), \quad 0 < r < 1 ,
$$

(1.1)

$$
\lim_{r \to 0^+} p(r)G(u'(r)) = 0, \quad p(1)G(u'(1)) = C.
$$

(1.2)

This is a mathematical model for many practical applications. In particular, we discuss here a classical problem in the capillarity theory. So, let $u = u(x, y)$ denote the height of a liquid in a vertical tube above the reference plane $u = 0$. Then the equilibrium shape of the liquid surface with constant surface tension in a uniform gravity field is described by the classical equation of capillarity [6], [7, pp. 262-263], [8, pp. 289-293]

$$
div \frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} = ku, \quad (x, y) \in \Omega,
$$

(1.3)

where $\Omega$ is the domain of the plane $u = 0$ occupied by the tube and $k$ is a positive constant. The natural physical boundary condition associated to eq. (1.3) is

$$
(1 + |\nabla u|^2)^{-1/2} \frac{\partial u}{\partial n} = \cos \gamma,
$$

(1.4)

where $\gamma$ is the contact angle (i.e., the angle between the liquid surface and the lateral surface of the tube) and $n$ is the outward normal to the boundary $\partial \Omega$.

Now, we consider the functional (see [2] for $\Omega = R^N$)

$$
J(v) = \int_{\Omega} \{(1/2)\Phi(\|\nabla v\|^2) + h(v)\} dx - \int_{\partial \Omega} \phi v \, dx ,
$$

(1.5)

where $\Omega$ is a domain of $R^N$, $N \geq 2$. It is easy to see that formally every critical point $u$ of $J$ (i.e., $J'(u) = 0$) is a solution of the following BVP

$$
div(\Phi'(\|\nabla u\|^2)\nabla u) = h'(u) , \quad in \, \Omega ,
$$

(1.6)

$$
\Phi'(\|\nabla u\|^2) \frac{\partial u}{\partial n} = \phi , \quad on \, \partial \Omega .
$$

(1.7)
This BVP contains as a particular case the problem (1.3), (1.4). Indeed, taking

\[ N = 2, \quad \Phi(\xi) = 2((\xi + 1)^{1/2} - 1), \quad \phi = \cos \gamma, \quad h(\xi) = k\xi^2 / 2, \]

we easily observe that (1.6), (1.7) coincides with the capillarity problem (1.3), (1.4). In this paper we restrict ourselves to the particular case in which \( \Omega \) is the unit sphere

\[ B(0, 1) = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \| x \|^2 = x_1^2 + \cdots + x_N^2 < 1 \} \]

and \( u \) depends only on \( r = \| x \| \) (so that it is natural to suppose that \( \phi \equiv C = \text{Const.} \)). In the case of the capillarity problem (1.3), (1.4) this means that the tube is circular and the height of the liquid \( u \) depends only on the distance \( r \) from the origin (in other words, \( u \) is radially symmetric). This is not a restriction because, as it is proved by R. Finn [6], the solution of (1.3), (1.4) with \( \Omega = B(0, 1) \) is unique and it coincides with the radially symmetric solution whose existence will be proved here. On the other hand, from a physical point of view, it is natural to suppose that \( u \) is radially symmetric because in the case of a circular tube the liquid surface is a rotation surface.

Now, assuming as we said above, that \( \Omega \) is the unit sphere of \( \mathbb{R}^N, \ N \geq 2 \) and that \( u = u(r) \), we can pass in (1.6), (1.7) to spherical coordinates: \( x_1 = r\cos \theta_1, \ x_2 = r\sin \theta_1 \cos \theta_2, \ x_3 = r\sin \theta_1 \sin \theta_2 \cos \theta_3, \ldots, x_{N-1} = r\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{N-2} \cos \theta_{N-1}, \ x_N = r\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{N-2} \sin \theta_{N-1}, \ 0 \leq r \leq 1, \ 0 \leq \theta_1 \leq \pi, \ldots, 0 \leq \theta_{N-2} \leq \pi, \ 0 \leq \theta_{N-1} \leq 2\pi \). We easily get

\[ (r^{N-1}u'(r)\Phi'(u'(r)^2))' = r^{N-1}h'(u(r)), \ 0 < r < 1, \]

\[ u'(1)\Phi'(u'(1)^2) = C \]  

and also, after standard computations [5, p.366], the corresponding functional \( J \) takes the form

\[ J(v) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^1 r^{N-1} ((1/2)\Phi'(v'(r)^2) + h(v(r))) \, dr - Cv(1). \]  

Denoting

\[ j(\xi) = (1/2)\Phi(\xi^2), \ G(\xi) = j'(\xi) = \xi\Phi'(\xi^2), \ \Psi = \frac{\Gamma(N/2)}{2\pi^{N/2}} J \text{ and } H = h' \]

we can write (1.8), (1.9) and (1.10) as follows

\[ (r^{N-1}G(u'(r)))' = r^{N-1}H(u(r)), \ 0 < r < 1, \]

\[ G(u'(1)) = C, \]

\[ \Psi(v) = \int_0^1 r^{N-1} \{ j(v(r)) + h(v(r)) \} \, dr - Cv(1). \]

Let us remark that the problem of the existence of the solution of (1.6), (1.7) is not a trivial one. Indeed, we can see that even in the particular case of the capillarity problem (1.3), (1.4) the corresponding functional \( J \) is coercive at most on the space \( W^{1,1}(\Omega) \) which is not reflexive. However, in the 1-dimensional case (1.11), (1.12) we are able to
prove the existence and uniqueness under very general assumptions. We mention that
the problem (1.11), (1.12) in the case \( N = 2 \) and \( H(\xi) = \xi \) has been investigated in [3,
10, 12]. A little more general problem is considered in [4, 11], namely

\[
(r^a G(u'(r)))' = t^b u(r), \quad 0 < r < 1, \tag{1.14}
\]

\[
\lim_{r \to 0+} r^a G(u'(r)) = 0, \quad G(u'(1)) = C, \tag{1.15}
\]

where \( a, b \) are real constants satisfying \( b + 1 > \max(0, a) \).

We observe that an additional boundary condition at \( t = 0 \) is present in (1.15). As we shall see later this condition is superfluous if \( a > 0 \) as it is the case with eq. (1.11). As in [13, 14] we consider in this paper the more general BVP (1.1), (1.2). The
main result of this paper, Theorem 1.1 below, states the existence of a unique solution
\( u \in C^1[0, 1] \) for problem (1.1), (1.2) under the assumptions (H.1)-(H.4) below.

\begin{align*}
\text{(H.1)} & \quad G : R \to R \text{ is continuous, strictly increasing, and } G(0) = 0; \\
\text{(H.2)} & \quad p \in C(0, 1), \quad p(r) > 0 \text{ for all } r \in (0, 1), \quad q \in L^1(0, 1), \quad q(r) > 0 \text{ a.e. in } (0, 1), \text{ and} \\
& \quad \lim_{r \to 0+} \frac{1}{p(r)} \int_0^r q(s) \, ds = 0; \\
\text{(H.3)} & \quad H : R \to R \text{ is continuous, strictly increasing, } H(0) = 0, \text{ and} \\
& \quad p(1)H(\infty) > CC_1 \text{ if } C > 0, \text{ and } p(1)H(-\infty) < CC_1 \text{ if } C < 0, \tag{1.16}
\end{align*}

where \( C_1 = \min\{p(r)/\int_0^r q(s) \, ds; \ r \in (0, 1]\}; \)

\begin{align*}
\text{(H.4)} & \quad \text{If } z : [0, 1] \to R_+ \text{ is Lipschitz continuous and nondecreasing, then the} \\
& \quad \text{application} \\
& \quad r \to (1/p(r)) \int_0^r q(s)z(s) \, ds \\
& \quad \text{is nondecreasing.}
\end{align*}

The main result of this paper is

\textbf{Theorem 1.1} If assumptions (H.1)-(H.4) are satisfied, then problem (1.1), (1.2) has
a unique solution \( u \in C^1[0, 1] \).

In order to prove Theorem 1 we can use either a direct method or a variational
approach.

The proof of the main result

We shall sketch the proof of Theorem 1.1 using the direct method as developed in [13,
14].

\textbf{Lemma 2.1} If (H.1)-(H.3) are fulfilled and \( u \in C^1[0, 1] \) is a solution of equation (0.1),
then the following implications hold:

\[
u(0) = 0 \Rightarrow \text{either } u' \geq 0 \text{ in } [0, 1] \text{ or } u' \leq 0 \text{ in } [0, 1]; \tag{2.1}
\]
\( u(0) > 0 \Rightarrow u' > 0 \text{ in } [0, 1]; \tag{2.2} \)
\( u(0) < 0 \Rightarrow u' < 0 \text{ in } [0, 1]. \tag{2.3} \)

**Proof.** We multiply eq. (1.1) by \( u(r) \) and then integrate on \([0, r]\):

\[
p(r)u(r)G(u'(r)) = \int_0^r \left[ p(s)u'(s)G(u'(s)) + q(s)u(s)H(u(s)) \right] ds.
\]

Thus, by our hypotheses, it follows that

\[
\{ r \in (0, 1]; u(r) = 0 \} = \{ r \in (0, 1]; u'(0) = 0 \}.
\]

Moreover, this set is either an empty set or an interval \((0, \delta]\) and so we can easily derive (2.1), (2.2) and (2.3). Q.E.D.

**Proof of Theorem 1.1**

**The uniqueness.** If \( u_1, u_2 \in C^1[0, 1] \) are two solutions of (1.1), (1.2) then we can easily get

\[
\int_0^1 \{(u_1 - u_2)'[G(u_1' - G(u_2')]p(s) + (u_1 - u_2)[H(u_1) - H(u_2)]q(s)\} \, ds = 0.
\]

By the monotonicity of \( G \) and \( H \), this implies that \( u_1 = u_2 \) in \([0, 1]\).

**The existence.** We first remark that (1.1), (1.2) is equivalent with the following problem

\[
p(r)G(u'(r)) = \int_0^r q(s)H(u(s)) \, ds, \tag{2.4}
\]
\[u'(1) = \beta, \tag{2.5}\]

where \( \beta \in \mathbb{R} \) is such that \( G(\beta) = C/p(1) \).

In what which follows we shall discuss only the case \( C > 0 \), i.e. \( \beta > 0 \) (the case \( C < 0 \) is similar and for \( C = 0 \) our problem has the null solution). Let us associate to eq. (2.4) the initial condition

\[
u(0) = u_0. \tag{2.6}\]

For the time being we suppose in addition that

\[ G(\pm \infty) = \pm \infty, \quad H(\pm \infty) = \pm \infty, \quad F := G^{-1} \text{ and } H \text{ are Lipschitz continuous.} \tag{2.7} \]

Denoting \( y := u' \), the problem (2.4), (2.6) can be equivalently expressed as:

\[
y(r) = F\left( \frac{1}{p(r)} \int_0^r q(s)H(u_0 + \int_0^s y(t) \, dt) \, ds \right). \tag{2.8}\]

According to \((H.2)\) eq. (2.8) is well defined at \( r = 0 \). We continue with two auxiliary results, Lemma 2.2 and Lemma 2.3.
Lemma 2.2. If (H.1) – (H.4) and (2.7) hold, then for each \( u_0 \in R \) eq. (2.8) has a unique solution \( y = y(r, u_0) \in C[0,1] \).

**Proof of Lemma 2.2.** We can apply the Fixed Point Banach’s Principle for the operator \( T : C[0,1] \rightarrow C[0,1] \), \( (Ty)(r) = \) the right hand side of eq. (2.8). Indeed, it is easy to see that \( T \) is a contraction on \( C[0,1] \) endowed with the metric

\[
d(y_1, y_2) := \sup \{ e^{-Lt} |y_1(t) - y_2(t)| ; 0 \leq t \leq 1 \},
\]

where \( L > 0 \) is large enough. Q.E.D.

Lemma 2.3. If (H.1) – (H.4) and (2.7) hold, then the application \( (r, u_0) \rightarrow y(r, u_0) \) has the following properties

\[
y(0, u_0) = 0, \text{ for every } u_0 \in R; \tag{2.9}
\]

\[
y(r, 0) = 0, \text{ for every } r \in [0,1]; \tag{2.10}
\]

\[
y(r, u_0) > 0, \text{ for } 0 < r \leq 1 \text{ and } u_0 > 0; \tag{2.11}
\]

\[
0 < u_0 < \bar{u}_0 \Rightarrow y(r, u_0) < y(r, \bar{u}_0) \text{ for } 0 < r \leq 1; \tag{2.12}
\]

there exists \( C > 0 \) such that

\[
|y(r, u_0) - y(r, \bar{u}_0)| \leq C|u_0 - \bar{u}_0|, \text{ for } u_0, \bar{u}_0 \in R, 0 \leq r \leq 1; \tag{2.13}
\]

\[
y(1, u_0) \rightarrow \infty \text{ as } u_0 \rightarrow \infty. \tag{2.14}
\]

The proof of this lemma can be done by elementary arguments [13]. We shall omit it.

**Proof of Theorem 1.1** (continued). In what follows we shall first eliminate the assumption that \( F \) and \( H \) are Lipschitz continuous. To this purpose we replace \( F \) and \( H \) by their Yosida approximates:

\[
F_\chi := (1/\chi) (I - J_\chi^F) = FJ_\chi^F, \quad H_\chi := (1/\chi) (I - J_\chi^H), \quad \chi > 0,
\]

where \( J_\chi^F = (I + \chi F)^{-1} \) [9, p.20]. As \( F_\chi, H_\chi, \chi > 0 \) are Lipschitz continuous there is (cf. Lemma 2.2) \( u_\chi \in C^1[0,1] \) such that

\[
u'_\chi(r) = F_\chi \left( \frac{1}{p(r)} \int_0^r q(s) H_\chi(u_\chi(s)) \, ds \right), \tag{2.15}
\]

\[
u'_\chi(1) = \beta. \tag{2.16}
\]

It is easy to see that \( u'_\chi \) is nondecreasing \( \Rightarrow \)

\[
0 = u'_\chi(0) \leq u'_\chi(r) \leq u'_\chi(1) = \beta, \text{ for } 0 \leq r \leq 1, \chi > 0. \tag{2.17}
\]

On the other hand, the set \( \{u_\chi; \chi > 0\} \) is bounded in \( C[0,1] \). Indeed, \( \{u_\chi(0); \chi > 0\} \) is bounded (otherwise (2.14) would imply that \( u'_\chi(1) = y(1, u_\chi(0)) = \beta \) is unbounded.
and consequently $u_\lambda(r) = u_\lambda(0) + \int_0^r u_\lambda'(s) \, ds$ is uniformly bounded. By the Arzelà-Ascoli Criterion we can deduce that $u_\lambda \to u$ in $C[0, 1]$, as $\lambda \to 0^+$, on a subsequence. By standard arguments, involving the properties of the Yosida approximate, we can pass to the limit in (2.15), (2.16) thus obtaining that the limit $u$ is a solution of (2.4), (2.5). Moreover, $u_\lambda' \to u'$ in $C[0, 1]$, as $\lambda \to 0^+$ and $0 \leq u' \leq \beta$.

The next step of the proof of Theorem 1.1 is to eliminate the assumption $G(\pm \infty) = \pm \infty$. To do this we replace $G$ by the function $\tilde{G}$ defined by

$$
\tilde{G}(\xi) = \begin{cases} 
\xi & \text{for } \xi < 0, \\
G(\xi) & \text{for } 0 \leq \xi \leq \beta, \\
\xi - \beta + G(\beta) & \text{for } \xi > \beta 
\end{cases}
$$

Of course, the new problem has a solution $u \in C^1[0, 1]$ which satisfies $0 \leq u' \leq \beta$ and hence $u$ is also a solution of (2.4), (2.5).

The final step is to eliminate the assumption $H(\pm \infty) = \pm \infty$. We first observe that (H.3) implies that $C_1 \in (0, \infty)$ and there exists a positive constant $M$ such that $H(M) = \gamma C_1 / p(1) = G(\beta)C_1$. Let us denote $\gamma = M + \beta$ and let $\tilde{H} : R \to R$ be the function defined by

$$
\tilde{H}(\xi) = \begin{cases} 
\xi & \text{for } \xi < 0, \\
H(\xi) & \text{for } 0 \leq \xi \leq \gamma, \\
\xi - \gamma + H(\gamma) & \text{for } \xi > \gamma 
\end{cases}
$$

By the above part of the proof, the equation

$$
(p(r)G(u'(r)))' = q(r)\tilde{H}(u(r)), \quad 0 < r < 1
$$

with the condition $u'(1) = \beta$ has a unique solution $u \in C^1[0, 1]$ satisfying $0 \leq u' \leq \beta$. By the monotonicity of $G$, $\tilde{H}$ we can see that

$$
p(r)G(\beta) \geq p(r)G(u'(r)) = \int_0^r q(s)\tilde{H}(u(s)) \, ds
$$

$$
\geq \tilde{H}(u(0)) \int_0^r q(s) \, ds,
$$

for every $r \in (0, 1]$. Hence

$$
\tilde{H}(u(0)) \leq G(\beta)C_1 = H(M) = \tilde{H}(M).
$$

(2.20)

Since $\beta > 0$ it follows by Lemma 2.1 that $u(0) \geq 0$. Therefore, taking also into account (2.20), we have

$$
0 \leq u(0) \leq M.
$$

(2.21)

Using (2.21) and the fact that $0 \leq u' \leq \beta$ we can see that

$$
0 \leq u(r) = u(0) + \int_0^r u'(s) \, ds \leq \gamma.
$$
As \( H = \tilde{H} \) in \([0, \gamma]\) it follows that \( u \) is also a solution of (1.1), (1.2). The proof of Theorem 1.1 is now complete. Q.E.D.

The variational approach

Consider the space

\[
X := \{v; \quad q^{1/2}v \in L^2(0, 1), \quad p^{1/2}v' \in L^2(0, 1)\}.
\]

This is a Hilbert space with the scalar product

\[
<v_1, v_2>_{X} = \int_{0}^{1}(pv_1'v_2' + qv_1v_2)\,dr.
\]

Consider the functional \( \Psi : X \rightarrow R \) defined by

\[
\Psi(v) = \int_{0}^{1}[p(r)j(v'(r)) + q(r)h(v(r))]\,dr - Cv(1),
\]

where

\[
j(\xi) = \int_{0}^{\xi}G(s)\,ds, \quad h(\xi) = \int_{0}^{\xi}H(s)\,ds.
\]

Then our problem (1.1), (1.2) can be seen as the Euler-Lagrange equations for the minimization problem

\[
(P) \quad \text{Min}\{\Psi(v); \quad v \in X\}.
\]

Using the arguments of [10] we can see that \((P)\) has a solution \( u \in C^1[0, 1] \) which is the unique solution of our problem (1.1), (1.2). We do insist more on this variational approach which leads us to the result given by Theorem 1.1.

3. Final comments and generalizations

Remark 3.1. As observed in the proof of Theorem 2.1, we have \( u'(0) = 0 \). We ask ourselves about the relationship between the problem (1.1), (1.2) and the problem made up by eq. (1.1) and the boundary conditions

\[
u'(0) = 0, \quad p(1)G(u'(1)) = C.
\]

We remember that if \( u \in C^1[0, 1] \) is the solution of (1.1), (1.2) then \( u \) satisfies also (1.1), (3.1). The converse implication is not true as it is shown by the following counterexample [13]:

\[
p(r) = r^{-1}, \quad q(r) \equiv 1, \quad H(\xi) = G(\xi) = \xi.
\]

In this case the general solution of eq. (1.1) is given by

\[
u(r) = r[c_1I_{2/3}(2r^{3/2}/3) + c_2I_{-2/3}(2r^{3/2}/3)],
\]
where $I_{2/3}(r)$ represents the modified Bessel function of the first species and of index $2/3$. It is easy to see that $u$ given by (3.2) satisfies the condition $u'(0) = 0$. It follows that the problem (1.1), (3.1) has an infinite number of solutions while the problem (1.1), (1.2) has a unique solution. However, in certain situations the two problems are equivalent. For instance, this is the case if $p(r) = r^a$, $a > 0$ and $q, G, H$ satisfy $(H.1) - (H.4)$.

**Remark 3.2.** There are examples showing that the assumption (1.16) is "almost" minimal. Indeed, for

$$p(r) = 1, \quad q(r) = 1, \quad G(\xi) = \xi, \quad H(\xi) = (1/\pi)\arctan \xi, \quad \text{and} \quad \beta = 1$$

(1.16) is not verified and in this case problem (1.1), (1.2) has no solution (see [14]).

**Remark 3.3.** An inspection of the proof of Theorem 1.1 shows that, in fact, for $\beta > 0$, we may consider $G : [0, \beta] \to R_+$ and $H : [0, \gamma] \to R_+$, where $G$ and $H$ are continuous, strictly increasing, $G(0) = H(0) = 0$, and $\gamma$ is such that

$$\gamma > \beta \quad \text{and} \quad C_1 G(\beta) \leq H(\gamma - \beta).$$

(3.3)

Indeed, $G$ and $H$ can be extended to $R$ some functions $\tilde{G}$ and $\tilde{H}$ (as in the proof of Theorem 1.1). By (3.3) there is $M \in (0, \gamma)$ such that $H(M) = \tilde{H}(M) = C_1 G(\beta)$ and so the corresponding solution of (1.1), (1.2) with $\tilde{G}$, $\tilde{H}$ instead of $G$, $H$ satisfies

$$0 \leq u(r) \leq M + \beta \quad \text{and} \quad 0 \leq u'(r) \leq \beta.$$ 

Hence $u$ is also a solution of (1.1), (1.2).

**Remark 3.4.** If in Theorem 1.1 $H$ is supposed to be only nondecreasing then the existence still holds with uniqueness up to an additive constant. To prove that we replace $H$ by $H_\epsilon = H + \epsilon I$, $\epsilon > 0$ which clearly is strictly increasing. We can show that the corresponding solution $u_\epsilon$ converges in $C^1[0, 1]$ to some $u$ which is a solution of (1.1), (1.2). In addition, any two solutions differ by an additive constant.

**Remark 3.5.** As in [10] we can prove the continuous dependence of the solution on $\beta$ (i.e., on $C$). An open problem is the continuous dependence on $p, q, G, H, \beta$. Another open problem is the generalization of problem (1.1), (1.2) to a Hilbert space, in the spirit of [1, p. 300]. Also, it remains to investigate the $N$-dimensional problem (1.6), (1.7) under reasonable assumptions, including the case of the capillarity in non-circular tubes.

**References**


