Quasilinear Elliptic Equations Involving Variable Exponents

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Abstract. Consider the boundary value problem \[-\sum_{i=1}^{N} \partial_{x_{i}} \left( a_{i}(x, \partial_{x_{i}} u) \right) \lambda(x) |u|^{q(x)-2} u = \lambda(x) u|^{q(x)-2} u \text{ in } \Omega, \]
\[u = 0 \text{ on } \partial \Omega, \]
where \(\Omega \subset R^{N}\) is a bounded domain with smooth boundary \(\partial \Omega\), the functions \(a_{i}(x, t)\) are of the type \(|t|^{p(x)-2} t\) with \(p_{i} \in C(\overline{\Omega})\), \(p_{i}(x) \geq 2\) for all \(x \in \Omega\), \(i = 1, \ldots, N\), \(q \in C(\overline{\Omega})\), \(\min_{\overline{\Omega}} q > 1\), and \(\lambda \in L^{\infty}(\Omega)\). In the particular case when \(a_{i}(x, t) = |t|^{p(x)-2} t\) for all \(i \in \{1, \ldots, N\}\), with \(p(x) \in C(\overline{\Omega})\) and \(\inf_{\Omega} p > 1\) the differential operator in (1) is the \(p(\cdot)\)-Laplace operator, i.e., \(\Delta_{p(x)} u := \text{div}(\nabla u|^{p(x)-2} \nabla u)\). Here, we allow different variable exponents for different spacial directions, \(a_{i}(x, t) = |t|^{p(x)-2} t\), so that we need an adequate functional framework for our problem, more precisely an anisotropic Sobolev space with variable exponents. In fact, in Theorem 1 below, which addresses existence and multiplicity of nontrivial weak solutions, even more general \(a_{i}\)'s are allowed, for instance \(a_{i}(x, t) = (1 + t^{2})|p(x)-2|/2 t\) with \(p_{i} \in C(\overline{\Omega})\) and \(\inf_{\Omega} p_{i} \geq 2\) for all \(i \in \{1, \ldots, N\}\).

It is worth mentioning that problem (1) (considered in the isotropic case, i.e. when all the functions \(a_{i}\) are equal) can serve as a model for phenomena which arise from the study of electrorheological fluids (see [3, 9, 14, 16, 19]), image processing (see [2]), or the theory of elasticity (see [22]). Finally, we note that the case \(a_{i}(x, t) = |t|^{m_{i}-2} t\) with \(m_{i} > 1\) positive constants was studied in [8].

MAIN RESULTS

In order to state our results we need to introduce the so-called

Sobolev spaces with variable exponents. Let us recall first the definitions of the spaces \(L^{p(\cdot)}(\Omega)\) and \(W^{1,p(\cdot)}_{0}(\Omega)\), where \(\Omega\) is a bounded domain in \(R^{N}\). We will also introduce an adequate functional space for problems of type (1).

Set \(C_{+}(\overline{\Omega}) = \{ h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 1\}\). For \(h \in C_{+}(\overline{\Omega})\) we set
\[h^{+} = \sup_{x \in \Omega} h(x), \quad h^{-} = \inf_{x \in \Omega} h(x).\]

For \(p \in C_{+}(\overline{\Omega})\), we define the so-called variable exponent Lebesgue space
\[L^{p(\cdot)}(\Omega) = \{ u : u \text{ is a measurable real–valued function such that } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \}.\]
endowed with the Luxemburg norm

\[ |u|_{p(\cdot)} = \inf \left\{ \mu > 0; \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}, \]

which is a separable and reflexive Banach space. If \( p_1, p_2 \in C_+([\Omega]) \) such that \( p_1 \leq p_2 \) in \( \Omega \), then the embedding \( L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega) \) is continuous.

Now, we define \( W_0^{1, p(\cdot)}(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) under the norm

\[ \|u\|_{1, p(\cdot)} = |\nabla u|_{p(\cdot)}. \]

We point out that the above norm is equivalent with the following norm

\[ \|u\|_{p(\cdot)} = \sum_{i=1}^N |\partial_i u|_{p(\cdot)}, \]

provided that \( p(x) \geq 2 \) for all \( x \in [\Omega] \) (see [11]). Hence \( W_0^{1, p(\cdot)}(\Omega) \) is a separable, reflexive Banach space. Note that if \( s \in C_+([\Omega]) \) and \( s(x) < p^*(x) \) for all \( x \in [\Omega] \), where \( p^*(x) = Np(x)/[N - p(x)] \) if \( p(x) < N \) and \( p^*(x) = \infty \) if \( p(x) \geq N \), then the embedding \( W_0^{1, p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega) \) is compact. For details on variable exponent Lebesgue and Sobolev spaces we refer to the book of Musielak [15] and the papers of Kováčik and Rákosník [10], Edmunds et al. [4, 5, 6], Samko and Vakulov [20].

Now, we introduce a natural generalization of the variable exponent Sobolev space \( W_0^{1, p(\cdot)}(\Omega) \) that will serve as our functional framework for problem (1). Let us denote by \( \overline{p} : [\Omega] \to \mathbb{R}^N \) the vectorial function \( \overline{p} = (p_1, \ldots, p_N) \). We define \( W_0^{1, \overline{p}(\cdot)}(\Omega) \), the anisotropic Sobolev space with variable exponents, as the closure of \( C_0^\infty(\Omega) \) with respect to the norm

\[ \|u\|_{\overline{p}(\cdot)} = \sum_{i=1}^N |\partial_i u|_{\overline{p}(\cdot)}. \]

As it was pointed out in [13], \( W_0^{1, \overline{p}(\cdot)}(\Omega) \) is a reflexive Banach space.

We also note that in the case when \( p_i \) are constant functions, the resulting anisotropic Sobolev space is denoted by \( W_0^{1, \overline{p}}(\Omega) \), where \( \overline{p} \) is the constant vector \( (p_1, \ldots, p_N) \). The theory of such spaces was developed by several authors, including [17, 18, 21].

We introduce \( \overline{p}_+, \overline{p}_- \in \mathbb{R}^N \) as

\[ \overline{p}_+ = (p_1^+, \ldots, p_N^+), \quad \overline{p}_- = (p_1^-, \ldots, p_N^-), \]

and \( P_+^+, P_-^+, P_-^- \in \mathbb{R}^+ \) as

\[ P_+^+ = \max\{p_1^+, \ldots, p_N^+\}, \quad P_-^- = \max\{p_1^-, \ldots, p_N^-\}, \quad P_-^- = \min\{p_1^-, \ldots, p_N^-\}. \]

Throughout this paper we assume that \( \sum_{i=1}^N \frac{1}{p_i} > 1 \), and define \( P_+^- \in \mathbb{R}^+ \) and \( P_-^+ \in \mathbb{R}^+ \) by

\[ P_+^- = \frac{N}{\sum_{i=1}^N 1/p_i - 1}, \quad P_-^+ = \max\{P_+^-, P_-^+\}. \]

We recall that if \( s \in C_+([\Omega]) \) satisfies \( 1 < s(x) < P_-^+ \) for all \( x \in [\Omega] \), then the embedding \( W_0^{1, \overline{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega) \) is compact (see [13, Theorem 1]).

**Existence and multiplicity of solutions.** By a weak solution to problem (1) we mean a function \( u \in W_0^{1, \overline{p}(\cdot)}(\Omega) \) such that

\[ \int_{\Omega} \left\{ \sum_{i=1}^N a_i(x, \partial_i u)\partial_i \phi - \lambda(x)|u|^{q(x)-2}u\phi \right\} \, dx = 0 \]

for all \( \phi \in W_0^{1, \overline{p}(\cdot)}(\Omega) \).
The following assumptions are required:

- \( a_i(x,t) \) satisfy some growth and convexity conditions, so that they are of the type \(|t|^{p_i(x)-2}t\) with \( p_i \in C(\overline{\Omega}) \), \( p_i(x) \geq 2 \) for all \( x \in \overline{\Omega}, i = 1, \ldots, N \).
- Function \( \lambda \) vanishes on a closed shell \( \subset \Omega \) (for example, the region between two concentric spheres \( B_r(x_0) \) and \( B_R(x_0) \), \( 0 < r < R \)), while \( \lambda \) is positive outside of the shell.
- \( q \in C(\overline{\Omega}) \) and \( 1 \leq q(x) < P_{w-} \) for all \( x \in \overline{\Omega} \).
- either \( \max_{\overline{B_r(x)}} q < P_- \leq P_{w-} < \min_{\Omega,B_r(x)} q \), or, \( \max_{\Omega,B_r(x)} q < P_- \leq P_{w-} < \min_{\overline{B_r(x)}} q \).

**Theorem 1.** Under the above assumptions there exists a \( \lambda^* > 0 \) such that problem (1) has two positive nontrivial solutions for each \( \lambda \) with \( |\lambda|_{L^p(\Omega)} < \lambda^* \).

**The eigenvalue problem.** Here we assume that \( \lambda \) is a positive constant. Denote by \( w_i \) the width of \( \Omega \) in the \( e_i \) direction, i.e., \( w_i = \sup_{x \in \Omega} \{x - y \cdot e_i\} \).

Our assumptions on \( a_i, q, p_i \) will be the following:

- \( a_i(x,t) = |t|^{p_i(x)-2}t \) with \( p_i \in C(\overline{\Omega}) \), \( p_i(x) \geq 2 \) for all \( x \in \overline{\Omega}, i = 1, \ldots, N \).
- there exists a \( j \in \{1, \ldots, N\} \) such that \( q(x) = q(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \) (i.e. \( q \) is independent of \( x_j \)) and \( p_j(x) = q(x) \) for all \( x \in \overline{\Omega} \). Moreover, \( q(x) \leq \frac{1}{w_i} + 1, \forall x \in \overline{\Omega} \).
- there exists a \( k \in \{1, \ldots, N\} \) (\( k \neq j \) above) such that \( \sup_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p_k(x) \).

Define

\[
\lambda_1 = \inf_{u \in W_0^{1,p_j}(\Omega),\{0\}} \frac{\int_{\Omega} \sum_{j=1}^{N} \frac{1}{p_j(x)} |\partial_j u|^{p_j(x)} \, dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx},
\]

\[
\lambda_0 = \inf_{u \in W_0^{1,p_j}(\Omega),\{0\}} \frac{\int_{\Omega} \sum_{j=1}^{N} |\partial_j u|^{p_j(x)} \, dx}{\int_{\Omega} |u|^{q(x)} \, dx}.
\]

**Theorem 2.** In addition to the above conditions, assume \( q(x) < P_- \) for all \( x \in \overline{\Omega} \). Then \( 0 < \lambda_0 \leq \lambda_1 \) and every \( \lambda \in (\lambda_1, \infty) \) is an eigenvalue of problem (1), while no \( \lambda \in (0, \lambda_0) \) can be an eigenvalue of problem (1).

**REFERENCES**