REGULARITY OF SOLUTIONS OF THE TELEGRAPH SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

S. AIZICOVICI
Ohio University, Department of Mathematics, Athens, Ohio 45701, USA

G. MOROSANU*
University of Iasi, Faculty of Mathematics, Iasi 6600, Romania

N. H. Pavel
Ohio University, Department of Mathematics, Athens, Ohio 45701, USA

Dedicated to Professor C. Corduneanu on the occasion of his 70th birthday

Communicated by the Editors

Abstract. We study the regularity of solutions to a nonlinear boundary value problem for the telegraph system. The main tools of our approach are a D'Alembert type representation formula for the solutions, and the Contraction Mapping Principle.

Key words. Telegraph system, nonlinear boundary value problem, regularity.

AMS Subject Classification. 35L50.

1. Introduction. Consider the following boundary value problem (BVP):

\[
\begin{align*}
\begin{cases}
    u_t + u_x + ru &= f_1(x,t), \\
    v_t + u_x + gv &= f_2(x,t), \\
    r_0 u(0,t) + v(0,t) &= 0, \\
    u(1,t) &= \beta(v(1,t)), \\
    u(x,0) &= u_0(x), \\
    v(x,0) &= v_0(x),
\end{cases}
\end{align*}
\tag{S}
\]

where \( r_0 > 0, r \geq 0, g \geq 0 \) are given constants and \( \beta \) is a given nonlinear function from \( \mathbb{R} \) into \( \mathbb{R} \). This BVP is a model for electrical or electronic circuits, with a nonlinear resistance at the end \( x = 1 \). In the case of integrated circuits, we have a system of \( 2n \) equations instead of (S), but the corresponding BVP has essentially the same behavior. For applications of this BVP we refer, e.g., to [4] and [12] (see also the references therein). An existence theory for BVP in the case when \( \beta \) is monotone (i.e. \( \beta \) is a nondecreasing function) can be found in [8, Chapter 3]. If \( \beta \) is a maximal monotone function (for instance, nondecreasing and continuous), and \( f_1, f_2, u_0, v_0 \) are smooth functions such that \( u_0, v_0 \) satisfy the zero-th order compatibility conditions

\* This paper was completed while the second author was a visiting professor at Ohio University.
(i.e., \( r_0 u_0(0) + v_0(0) = 0, u_0(1) = \beta(v_0(1)) \)), then BVP has a unique strong solution (see [8, Chapter 3]).

The aim of this paper is to give some results concerning the higher order regularity of solutions to BVP. As we shall see, this requires higher order compatibility conditions for \( u_0, v_0, f_1, f_2 \). The main difficulty of this problem comes from the nonlinearity of (BC). Notice in addition that our BVP is a hyperbolic problem and so no smoothing effect on the data is expected.

The regularity question for BVP is important in itself but is also appears as an intermediate step in developing an asymptotic analysis for this problem. Specifically, let us assume that the term \( u_t \) in (S) has a small coefficient \( \varepsilon > 0 \). In practice, this means that the inductance of the electrical circuit is a small parameter. If we put \( \varepsilon = 0 \), as engineers usually do, then our BVP becomes a parabolic BVP in \( v \), provided \( r > 0 \) (see, e.g., [9]). So, our original hyperbolic BVP is drastically changed. On the other hand, \( \varepsilon \) is just small not null. A natural question that arises is whether the parabolic reduced BVP still describes properly the same phenomenon (i.e. the solution of the original BVP, with \( \varepsilon u_t \) instead of \( u_t \), is "close enough" to the solution of the reduced parabolic model). A very efficient method to see what happens is the so-called boundary layer function method due to M.I. Vishik and L.A. Lyusternik (see, e.g., [14]). This method is based on asymptotic expansions of the solution of BVP (with \( \varepsilon u_t \) instead of \( u_t \)), including some correctors (or boundary layer functions). A first step in applying the method of Vishik and Lyusternik is to find heuristically the correctors and indicate the problems verified by the other terms of the expansion. For instance, the remainder terms satisfy a BVP similar to our original BVP (with \( \varepsilon u_t \) instead of \( u_t \)).

In order to validate the asymptotic expansion we need estimates for the remainder terms. But this requires, among other things, higher order regularity of the remainder terms. This is a strong motivation for our present paper.

Notice that the method of Vishik and Lyusternik has already been used in [1], [2], [3], and [10] for the case of non-local boundary conditions of the form:

\[
\begin{align*}
&\left\{ \\
&\quad \begin{array}{c}
& r_0 u(0, t) + v(0, t) = 0, \\
& u(1, t) = c v_t(1, t) + \beta(v(1, t)), \\
& t > 0 \quad (c > 0).
\end{array}
\end{align*}
\]

Surprisingly, the regularity problem in that case was easier than in the present case, due to an appropriate functional framework used there (which does not fit here).

Our paper is organized as follows: In Section 2 we recall known existence and regularity results. In Section 3, the regularity problem is solved in the particular case \( r = g = 0 \) by a D'Alambert type technique. In Section 4 the general case is solved by using an appropriate fixed point method. Some comments and generalizations are discussed in Section 5.
2. Review of known existence and regularity results.

We start with the following existence and regularity result

**Theorem 2.1.** Assume that $\beta$ is a maximal monotone mapping from $\mathbb{R}$ into $\mathbb{R}$ (for simplicity, $\beta$ is assumed to be single-valued with $D(\beta) = \mathbb{R}$); $f_1, f_2 \in W^{1,1}(0, T; L^2(0, 1))$, $T > 0$ fixed; $u_0, v_0 \in H^1(0, 1)$ and satisfy the following (zero-th order) compatibility conditions

\[
\begin{cases}
   r_0 u_0(0) + v_0(0) = 0, \\
   u_0(1) = \beta(v_0(1)).
\end{cases}
\tag{2.1}
\]

Then, the problem \((S), (BC), (IC)\) has a unique solution \((u, v) \in W^{1,\infty}(0, T; L^2(0, 1))^2\), with $u_x, v_x \in L^\infty(0, T; L^2(0, 1))$.

The proof of Theorem 2.1 can be done by using the theory of evolution equations associated to monotone operators. Precisely, our BVP is expressed as a Cauchy problem for an ordinary differential equation in the space $H = L^2(0, 1)^2$ (see [8, Chapter 3]). So, the solution given by Theorem 2.1 is a strong solution of this Cauchy problem, in the usual sense (as defined, e.g., in [8, Chapter 1]). In addition to the regularity properties given by the general theory, in our case we get $u_x, v_x \in L^\infty(0, T; L^2(0, 1))$. Let us also remark that conditions (BC) are verified at every $t \in [0, T]$. As our problem is a hyperbolic one, the compatibility conditions (2.1) are essential for the existence of a strong solution.

Notice also that, in particular, the solution \((u, v)\) given by Theorem 2.1 above, satisfies

\[u, v \in H^1(0, T; L^2(0, 1)) \cap H^1(0, 1; L^2(0, T)) = H^1(Q_T),\]

where $Q_T = (0, 1) \times (0, T)$. We know that $H^1(Q_T)$ is not included in $C(\overline{Q}_T)$ and so, in spite of the additional regularity properties, it seems that $u, v$ do not belong to $C(\overline{Q}_T)$. However, $u, v \in L^\infty(Q_T)$ as can be easily seen by the formula

\[u(x, t) = \int_0^1 [yu_y(y, t) + u(y, t)]dy - \int_x^1 u_y(y, t)dy.\]

Finally, let us recall the following result concerning the existence of weak solutions:

**Theorem 2.2.** If $\beta$ is as above; $f_1, f_2 \in L^1(0, T; L^2(0, 1))$; and $u_0, v_0 \in L^2(0, 1)$, then our BVP has a unique weak solution \((u, v) \in C([0, T]; L^2(0, 1))^2\) (i.e., \((u, v)\) is a limit in $C([0, T]; L^2(0, 1))^2$ of strong solutions given by Theorem 2.1 above).

3. Regularity in the case $r = g = 0$. Throughout this section we suppose that $r = g = 0$. So we can take advantage of the fact that in this particular case \((S)\) has an explicit general solution. This is not true for any coefficients in \((S)\) (see [7, p. 58]). For the sake of simplicity, we restrict our attention to proving that, under appropriate assumptions, \((u, v)\) is of class $C^1(\overline{Q}_T)^2$.

The main result in this section is
Theorem 3.1. Assume that $\beta \in C^1(\mathbb{R})$, $\beta' \geq 0$; $f_1, f_2 \in C^1(\bar{Q}_T)$; $u_0, v_0 \in C^1[0, 1]$ and satisfy (2.1). Moreover, the following first order compatibility conditions are satisfied
\[
\begin{align*}
  & r_0[f_1(0,0) - v_0(0)] + f_2(0,0) - u_0(0) = 0, \\
  & f_1(1,0) - v_0'(1) = \beta'(v_0(1)) \cdot [f_2(1,0) - u_0'(1)].
\end{align*}
\] (3.1)

Then, the solution $(u,v)$ of (S) (with $r = g = 0$), (BC), (IC) belongs to $C^1(\bar{Q}_T)^2$.

**Proof.** The general solution of (S) with $r = g = 0$ is given by the following formulas of D’Alembert’s type:
\[
\begin{align*}
  u(x,t) &= \frac{1}{2} \{ \varphi(x-t) + \psi(x+t) + \int_0^t [(f_1 + f_2)(x-s,t-s) \nonumber \\
  & \quad + (f_1 - f_2)(x+s,t-s)]ds \}, \\
  v(x,t) &= \frac{1}{2} \{ \varphi(x-t) - \psi(x+t) + \int_0^t [(f_1 + f_2)(x-s,t-s) \nonumber \\
  & \quad - (f_1 - f_2)(x+s,t-s)]ds \},
\end{align*}
\] (3.2)

where $\varphi, \psi$ are arbitrary functions of class $C^1$. We have considered in (3.2) that $f_1, f_2$ are extended for $x \in \mathbb{R}$, for instance in the following way:
\[
\begin{align*}
  f_1(x,t) &= -f_1(2-x,t) + 2f_1(1-0,t), \quad \text{for } 1 < x \leq 2, \\
  f_1(x,t) &= -f_1(-x,t) + 2f_1(0+0,t), \quad \text{for } -1 \leq x < 0,
\end{align*}
\]

and so on. Obviously, these extensions are in $C^1(\mathbb{R} \times [0,T])$. When necessary, any other function defined on $\bar{Q}_T$ will be extended similarly. Now, we are going to determine $\varphi$ and $\psi$ in (3.2) such that $u, v$ satisfy (IC) and (BC). Obviously, we need $\varphi$ in $[-T,1]$ and $\psi$ in $[0,1+T]$. By using (IC) we can determine $\varphi, \psi$ in $[0,1]:$
\[
\begin{align*}
  & \begin{cases}
  \varphi(x) = u_0(x) + v_0(x), \\
  \psi(x) = u_0(x) - v_0(x),
  \end{cases} \quad 0 \leq x \leq 1.
\end{align*}
\] (3.3)

Now, we require that $u, v$ given by (3.2) satisfy (BC):
\[
\begin{align*}
  & r_0[\varphi(-t) + \psi(t)] + \int_0^t [(f_1 + f_2)(-s,t-s) + (f_1 - f_2)(s,t-s)]ds \\
  & \quad + \varphi(-t) - \psi(t) + \int_0^t [(f_1 + f_2)(-s,t-s) - (f_1 - f_2)(s,t-s)]ds = 0,
\end{align*}
\] (3.4)

\[
\begin{align*}
  & \frac{1}{2} \{ \varphi(1-t) + \psi(1+t) + \int_0^t [(f_1 + f_2)(1-s,t-s) + (f_1 - f_2)(1+s,t-s)]ds \} \\
  & \quad = \beta(\frac{1}{2} \{ \varphi(1-t) - \psi(1+t) + \int_0^t [(f_1 + f_2)(1-s,t-s) \nonumber \\
  & \quad - (f_1 - f_2)(1+s,t-s)]ds \}, \quad 0 \leq t \leq T.
\end{align*}
\] (3.5)
Eq. (3.4) gives \( \varphi \) in \([-1, 0]\). By virtue of (2.1) and (3.4), \( \varphi(0 - 0) = \varphi(0 + 0) \), that is, \( \varphi \) is continuous on \([-1, 1]\). Now, from (3.5) we can find \( \psi \) uniquely in \((1, 2]\). Indeed, if we denote

\[
\begin{align*}
    h_1(t) := & \int_0^t (f_1 + f_2)(1 - s, t - s)ds, \\
    h_2(t) := & \int_0^t (f_1 - f_2)(1 + s, t - s)ds, \\
    z(t) := & \frac{1}{2} \{ \varphi(1 - t) - \psi(1 + t) + h_1(t) - h_2(t) \},
\end{align*}
\]

we can write (3.5) in the form

\[
z(t) + \beta(z(t)) = \varphi(1 - t) + h_1(t). \tag{3.6}
\]

So

\[
z(t) = (I + \beta)^{-1}(\varphi(1 - t) + h_1(t))
\]

and therefore

\[
\psi(1 + t) = \varphi(1 - t) + h_1(t) - h_2(t) - 2(I + \beta)^{-1}(\varphi(1 - t) + h_1(t)), \quad 0 < t \leq 1. \tag{3.7}
\]

Letting \( t \to 0^+ \) in (3.7) one gets

\[
\psi(1 + 0) = \varphi(1 - 0) - 2(I + \beta)^{-1}(\varphi(1 - 0)).
\]

This implies (see (2.1) and (3.3))

\[
\psi(1 + 0) = \psi(1 - 0),
\]

that is, \( \psi \in C[0, 2] \). So, \( \varphi \) can be determined from (3.4) on the interval \([-2, 0]\) and \( \varphi \in C[-2, 1] \). Continuing this procedure, we obtain uniquely \( \varphi \in C[-T, 1], \psi \in C[0, 1 + T] \). Actually, by (3.1), (3.4) and (3.5) it is easily seen that

\[
\varphi \in C^1[-T, 1] \quad \text{and} \quad \psi \in C^1[0, 1 + T].
\]

So, we may conclude that \((u, v) \in C^1(\bar{Q}_T)^2 \). \( \square \)

**Remark 3.1.** The compatibility conditions (3.1) are quite natural. Indeed, if \((u, v) \in C^1(\bar{Q}_T)^2 \) satisfies BVP, then we can differentiate (BC) and for \( t = 0 \) we necessarily obtain that \( u_0, v_0, f_1, f_2 \) satisfy (3.1). We have only to notice that

\[
\begin{align*}
    u_t(0, 0) &= f_1(0, 0) - v_x(0, 0) = f_1(0, 0) - v_0'(0), \\
    v_t(0, 0) &= f_2(0, 0) - u_0'(0), \\
    u_t(1, 0) &= f_1(1, 0) - v_0'(1), \\
    v_t(1, 0) &= f_2(1, 0) - u_0'(1).
\end{align*}
\]
As (3.1) guarantee the fact that $u, v \in C^1(\bar{Q}_T)$, it is natural to call them first order compatibility conditions.

**Remark 3.2.** If we assume only that $\beta \in C(\mathbb{R})$, $\beta$ monotone; $f_1, f_2 \in C(\bar{Q})$; $u_0, v_0 \in C[0, 1]$ and satisfy (2.1), then we can determine uniquely $\varphi \in C[-T, 1], \psi \in C[0, 1+T]$ from (3.2). In this case, the pair $(u, v) \in C(\bar{Q}_T)^2$ can be viewed as a weak solution of (S) (with $r = g = 0$), (BC), (IC).

4. $C^1$-regularity in the general case.

In order to extend Theorem 3.1 to the general case we shall use a fixed point technique. Of course, (3.1) must be replaced by the following first order compatibility conditions:

\[
\begin{align*}
& r_0[f_1(0, 0) - ru_0(0) - v_0(0)] + f_2(0, 0) - g v_0(0) - u'_0(0) = 0, \\
& f_1(0, 0) - ru_0(1) - v_0(1) = \beta'(v_0(1)) \cdot [f_2(1, 0) - g v_0(1) - u'_0(1)].
\end{align*}
\]

(4.1)

The main result of this section is the following

**Theorem 4.1.** Assume that $\beta \in W^{2, \infty}_{\text{loc}}(\mathbb{R})$, $\beta' \geq 0$; $f_1, f_2 \in C^1(\bar{Q}_T)$; $u_0, v_0 \in C[0, 1]$.
If also (2.1) and (4.1) are fulfilled, then our BVP has a unique solution $(u, v) \in C(\bar{Q}_T)^2$.

**Proof.** By Theorem 2.1 we know that BVP has a unique strong solution $(u, v)$. But we have to show that $(u, v) \in C^1(\bar{Q}_T)$. Consider the set

\[
Y = \{ (\alpha_1, \alpha_2) \in C^1(\bar{Q}_T)^2; \quad \alpha_1(0, 0) = u_0(0), \\
\alpha_1(1, 0) = u_0(1), \quad \alpha_2(0, 0) = v_0(0), \quad \alpha_2(1, 0) = v_0(1) \}.
\]

Then, by (4.1), Theorem 3.1 can be applied to conclude that for every $\alpha = (\alpha_1, \alpha_2) \in Y$ the system

\[
\begin{align*}
& u_t + u_x = f_1 - r \alpha_1, \\
& v_t + u_x = f_2 - g \alpha_2,
\end{align*}
\]

(4.2)

with (BC) and (IC) has a unique solution $(u_\alpha, v_\alpha) \in C^1(\bar{Q}_T)^2$. Let us fix some $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2) \in Y$ and denote $(\hat{u}, \hat{v}) := (u_{\hat{\alpha}}, v_{\hat{\alpha}})$ (i.e., the solution of (4.2), (BC), (IC) with $(\alpha_1, \alpha_2) = (\hat{\alpha}_1, \hat{\alpha}_2)$). In a first stage, we intend to prove our theorem for a small time interval, say $[0, T]$, but in order not to introduce other notations we consider that $T$ itself is small enough, $0 < T \leq 1$. Now, consider the set

\[
S = \{ \alpha = (\alpha_1, \alpha_2) \in Y; ||\alpha_1 - \hat{u}||_1 \leq b, ||\alpha_2 - \hat{v}||_1 \leq b \},
\]

where $b$ is a fixed positive number and $|| \cdot ||_1$ denotes the usual norm of $C^1(\bar{Q}_T)$. Define on $S$ an operator $B$ by

\[
B \left( \begin{array}{c}
\alpha_1 \\
\alpha_2
\end{array} \right) = \left( \begin{array}{c}
u_\alpha \\
v_\alpha
\end{array} \right), \quad (\forall) \alpha = (\alpha_1, \alpha_2) \in S.
\]
We intend to apply Banach's Fixed Point Principle. First of all, we notice that $S$ is a closed subset of $C^1(\bar{Q}_T)^2$ and so it is a metric space with the metric

$$d\left(\begin{vmatrix} \alpha_1 \\ \alpha_2 \end{vmatrix}, \begin{vmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{vmatrix}\right) = \max(\|\alpha_1 - \bar{\alpha}_1\|_1, \|\alpha_2 - \bar{\alpha}_2\|_1).$$

Let us show that $B$ maps $S$ into itself. So, take $\alpha = (\alpha_1, \alpha_2) \in S$. Clearly,

$$B\alpha = \begin{vmatrix} u_\alpha \\ v_\alpha \end{vmatrix} \in Y.$$

We have to prove that

$$\|u_\alpha - \hat{u}\|_1 \leq b \quad \text{and} \quad \|v_\alpha - \hat{v}\|_1 \leq b.$$

Of course, $(u_\alpha, v_\alpha)$ and $(\hat{u}, \hat{v})$ can be represented by formulas of the form (3.2), with some $(\varphi, \psi)$, $(\hat{\varphi}, \hat{\psi})$ and $(f_1 - r\alpha_1, f_2 - g\alpha_2)$, $(f_1 - r\hat{\alpha}_1, f_2 - g\hat{\alpha}_2)$ instead of $(f_1, f_2)$. Therefore,

$$\begin{split}(u_\alpha - \hat{u})(x, t) &= \frac{1}{2}\{((\varphi - \hat{\varphi})(x - t) + (\psi - \hat{\psi})(x + t) \\
&\quad - \int_0^t [r(\alpha_1 - \hat{\alpha}_1) + g(\alpha_2 - \hat{\alpha}_2)](x - s, t - s)ds \} \quad (4.3) \\
&\quad + \int_0^t [r(\alpha_1 - \hat{\alpha}_1) - g(\alpha_2 - \hat{\alpha}_2)](x + s, t - s)ds \},
\end{split}$$

$$\begin{split}(v_\alpha - \hat{v})(x, t) &= \frac{1}{2}\{((\varphi - \hat{\varphi})(x - t) - (\psi - \hat{\psi})(x + t) \\
&\quad - \int_0^t [r(\alpha_1 - \hat{\alpha}_1) + g(\alpha_2 - \hat{\alpha}_2)](x - s, t - s)ds \} \quad (4.4) \\
&\quad + \int_0^t [r(\alpha_1 - \hat{\alpha}_1) - g(\alpha_2 - \hat{\alpha}_2)](x + s, t - s)ds \}, 0 \leq t \leq T.
\end{split}$$

As $T$ is small, the integral terms of (4.3) and (4.4) satisfy the estimate:

$$\|\text{integral terms of (4.3), (4.4)}\|_{C^1(\bar{Q}_T)} \leq \delta d(\alpha, \hat{\alpha}), \text{ with } \delta \text{ small.} \quad (4.5)$$

So, it suffices to show that similar estimates take place for $(1/2)(\varphi - \hat{\varphi})(x - t)$ and $(1/2)(\psi - \hat{\psi})(x + t)$. Of course,

$$\begin{split}(\varphi - \hat{\varphi})(x - t) &= 0, \quad \text{for } x - t \geq 0, \\
(\psi - \hat{\psi})(x + t) &= 0, \quad \text{for } x + t \leq 1.
\end{split}$$
Otherwise, we can use the following equations

\[
(1 + r_0)(\varphi - \hat{\varphi})(-t) + (r_0 + 1) \int_0^t [-r(\alpha_1 - \hat{\alpha}_1) - g(\alpha_2 - \hat{\alpha}_2)](-s, t - s)ds \\
+ (r_0 - 1) \int_0^t [-r(\alpha_1 - \hat{\alpha}_1) + g(\alpha_2 - \hat{\alpha}_2)](s, t - s)ds = 0, \quad 0 \leq t \leq T;
\]

(here we have used \(\psi(t) = u_0(t) - v_0(t) = \hat{\psi}(t), \quad 0 \leq t \leq T\))

\[
\frac{1}{2} \{\psi(1 + t) + \varphi(1 - t) + \int_0^t (f_1 + f_2 - r\alpha_1 - g\alpha_2)(1 - s, t - s)ds \\
+ \int_0^t (f_1 - f_2 - r\alpha_1 + g\alpha_2)(1 + s, t - s)ds
\]

\[
= \beta\left(\frac{1}{2} \{\varphi(1 - t) - \psi(1 + t) + \int_0^t (f_1 + f_2 - r\alpha_1 - g\alpha_2)(1 - s, t - s)ds \\
- \int_0^t (f_1 - f_2 - r\alpha_1 + g\alpha_2)(1 + s, t - s)ds\}\right), \quad 0 \leq t \leq T,
\]

and a similar equation for \(\hat{\varphi}, \hat{\psi}\).

From (4.6) one can easily derive an estimate like (4.5) for \((1/2)(\varphi - \hat{\varphi})(x - t)\). On the other hand, from (4.7) we get

\[
z(t) + \beta(z(t)) = \varphi(1 - t) + h_1(t), \quad 0 \leq t \leq T,
\]

where

\[
z(t) := \frac{1}{2} \{\varphi(1 - t) - \psi(1 + t) + h_1(t) - h_2(t)\},
\]

\[
h_1(t) := \int_0^t (f_1 + f_2 - r\alpha_1 - g\alpha_2)(1 - s, t - s)ds,
\]

\[
h_2(t) := \int_0^t (f_1 - f_2 - r\alpha_1 + g\alpha_2)(1 + s, t - s)ds.
\]

A similar equation can be written for \((\hat{\varphi}, \hat{\psi})\):

\[
\hat{z}(t) + \beta(\hat{z}(t)) = \hat{\varphi}(1 - t) + \hat{h}_1(t), \quad 0 \leq t \leq T.
\]

Notice that

\[
\varphi(1 - t) = u_0(1 - t) + v_0(1 - t) = \hat{\varphi}(1 - t), \quad 0 \leq t \leq T.
\]
Using (4.8), (4.9), (4.10) and the monotonicity of \( \beta \), we can see that
\[
\sup_{Q_T} |(1/2)(\psi - \psi')(x + t)| \leq \delta d(\alpha, \dot{\alpha}), \quad \delta \text{ small.} \tag{4.11}
\]
Now, using again (4.8) and (4.9), one can obtain that
\[
[z'(t) - \dot{z}'(t)]^2[1 + \beta'(x)] \leq |\dot{z}'(t)| \cdot |z'(t) - \dot{z}'(t)|.
\]
\[
|\beta'(z(t)) - \beta'(\tilde{z}(t))| + |h_1(t) - \tilde{h}_1(t)| \cdot |z'(t) - \dot{z}'(t)|, \quad 0 \leq t \leq T. \tag{4.12}
\]
Therefore, as \( \beta' \) is Lipschitzian on bounded sets, we get from (4.12)
\[
\sup_{Q_T} |(1/2)(\psi' - \psi')(x + t)| \leq \delta d(\alpha, \dot{\alpha}), \quad \delta \text{ small.} \tag{4.13}
\]
Now, summarizing, we can see that
\[
\max(||u_{\alpha} - \dot{u}||_1, ||v_{\alpha} - \dot{v}||_1) \leq \delta d(\alpha, \dot{\alpha}), \quad \delta \text{ small.} \tag{4.14}
\]
Notice that we have denoted by \( \delta \) a generic small constant. Clearly, (4.14) implies that \((u_{\alpha}, v_{\alpha}) \in S\). Using the same reasoning as above, we can see that \( B \) is a contraction on \( S \) and so \( B \) has a unique fixed point in \( S \). This means that the solution of BVP is of class \( C^1(\bar{Q}_T)^2 \), for \( T_1 \) small, where \( Q_{T_1} = (0,1) \times (0,T_1) \). By using the same procedure, we can show the same thing on the interval \([T_1, 2T_1]\), starting with the initial data: \( u(\cdot, T_1), v(\cdot, T_1), f_1(x, t + T_1), f_2(x, t + T_1) \). After a finite number of steps the interval \([0, T]\) is completely covered. \( \square \)

5. Comments, extensions, and open problems.

5.1. We presented in the Introduction a motivation for our study related to a problem of asymptotic analysis in which the term \( u_t \) of (S) is replaced by \( \varepsilon u_t \), with \( \varepsilon > 0 \) small. Of course, all we have done before is still valid for this case with minor changes, including Theorem 4.1. In fact, this can be seen by a simple change of variable: \( \tau = t/\sqrt{\varepsilon} \).

5.2. In Theorem 4.1, \( \beta \) satisfies stronger assumptions than those used in Theorem 3.1. An open problem is the following: Does Theorem 4.1 still hold for \( \beta \in C^1(\mathbb{R}), \beta' \geq 0 \)? On the other hand, using the same fixed point procedure as before, we can prove the existence of a unique generalized solution to BVP, \((u, v) \in C(\bar{Q}_T)^2\), obtained as a fixed point of the operator \( B \), under the following weaker assumptions: \( \beta \in C(\mathbb{R}), \beta \) nondecreasing; \( f_1, f_2 \in C(\bar{Q}_T); u_0, v_0 \in C[0, 1] \) and satisfy (2.1). This generalized solution is stronger than the weak solution given by Theorem 2.2 above.

5.3. Our results can be extended, by using similar arguments, to time dependent boundary conditions, i.e., to the case when \( \beta = \beta(t, \xi) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \). More precisely, we have the following two results:
**Theorem 5.1.** Assume that $\beta = \beta(t, \xi) \in C([0, T] \times \mathbb{R})$, $\beta(t, \cdot)$ is nondecreasing for every $t \in [0, T]$; $f_1, f_2 \in C(\bar{Q}_T)$; $u_0, v_0 \in C(\bar{Q}_T)$ and satisfy the following zero-th order compatibility conditions:

$$
\begin{align*}
& r_0 u_0(0) + v_0(0) = 0, \\
& u_0(1) = \beta(0, v_0(1)).
\end{align*}
$$

(5.1)

Then, our BVP, with (BC) replaced by

$$
\begin{align*}
& r_0 u(0, t) + v_0(0, t) = 0, \\
& u(1, t) = \beta(t, v(1, t)), \quad 0 \leq t \leq T,
\end{align*}
$$

(BC)'

has a unique generalized solution $(u, v) \in C(\bar{Q}_T)^2$.

**Theorem 5.2.** Assume that $\beta = \beta(t, \xi) \in C^1([0, T] \times \mathbb{R})$, $\beta_{\xi} \geq 0$, the functions $\beta_t(t, \cdot)$, $\beta_{\xi}(t, \cdot)$ are uniformly Lipschitz on bounded sets; $f_1, f_2 \in C^1(\bar{Q}_T)$; $u_0, v_0 \in C^1[0,1]$ and satisfy (5.1) and the following first order compatibility conditions:

$$
\begin{align*}
& r_0 [f_1(0, 0) - ru_0(0) - v_0'(0)] + f_2(0, 0) - gv_0(0) - u_0'(0) = 0, \\
& f_1(1, 0) - ru_0(1) - v_0'(1) = \beta_t(0, v_0(1)) \\
& + \beta_{\xi}(0, v_0(1)) \cdot [f_2(1, 0) - gv_0(1) - u_0'(1)].
\end{align*}
$$

(5.2)

Then problem (S), (BC)', (IC) has a unique solution $(u, v) \in C^1(\bar{Q}_T)$.

The proofs of Theorems 5.1 and 5.2 follow the same steps as in the case when $\beta$ does not depend on $t$. It suffices to notice that if $\beta = \beta(t, \xi)$ satisfies the assumptions of Theorem 5.1 (Theorem 5.2) and $g = g(t) \in C[0, T]$ ($g \in C^1[0, T]$), the function $t \mapsto (I + \beta(t, \cdot))^{-1}g(t)$ belongs to $C[0, T]$ ($C^1[0, T]$, respectively).

**Remark 5.1.** If in Theorem 5.2, $\beta$ is a linear function with respect to $\xi$, i.e.,

$$
\beta(t, \xi) = a(t)\xi + b(t),
$$

then it suffices to assume that $a, b \in C^1[0, T]$, $a(t) \geq 0$ for $t \in [0, T]$.

**Remark 5.2.** Consider the space $H = L^2(0, 1)^2$, with the usual scalar product and norm. Define the operators $A(t) : D(A(t)) \subset H \to H$, $0 \leq t \leq T$, by

$$
D(A(t)) = \{(p, q) \in H^1(0, 1)^2; r_0 p(0) + q(0) = 0, p(1) = \beta(t, q(1))\},
$$

$$
A(t) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} q' + rp \\ p' + gq \end{pmatrix},
$$

where $\beta$ satisfies the assumptions of Theorem 5.2. So, problem (S), (BC)', (IC) can be expressed as the following initial value problem in $H$:

$$
\frac{d}{dt} \begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix} + A(t) \begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix} = \begin{pmatrix} f_1(\cdot, t) \\ f_2(\cdot, t) \end{pmatrix}, \quad 0 \leq t \leq T,
$$

(5.3)
\[
\begin{pmatrix}
u(\cdot, 0) \\
v(\cdot, 0)
\end{pmatrix}
= 
\begin{pmatrix}
u_0 \\
v_0
\end{pmatrix}.
\] (5.4)

It turns out that for each \( t \in [0, T] \), \( A(t) \) is a maximal monotone operator (see [8, Chapter III]). Clearly, the solution given by Theorem 5.2 is a strong solution of problem (5.3), (5.4), in the sense of the theory of evolution equations. We also have the following result, which seems to be new:

**Theorem 5.3.** Let \( \beta \) satisfy the same assumptions as in Theorem 5.2. If \( f_1, f_2 \in L^1(0, T; L^2(0, 1)) \) and \( u_0, v_0 \in L^2(0, 1) \), then problem (5.3), (5.4) has a unique weak solution \((u, v) \in C([0, T]; L^2(0, 1))^2 \) (i.e., \((u, v)\) is a limit in \( C([0, T]; L^2(0, 1))^2 \) of strong solutions).

**Proof.** First of all, notice that the set
\[
Z = \{(u_0, v_0, f_1, f_2) \in C^1[0, 1]^2 \times C^1(\bar{Q}_T)^2; u_0, v_0, f_1, f_2 \text{ satisfy (5.1) and (5.2)}\}
\]
is dense in \( L^2(0, 1)^2 \times L^1(0, T; L^2(0, 1))^2 \). On the other hand, if we denote by \((u, v), (\bar{u}, \bar{v}) \in C^1(\bar{Q}_T)^2 \) the solutions of problem (S) (BC)', (IC) corresponding to some \((u_0, v_0, f_1, f_2), (\bar{u}_0, \bar{v}_0, f_1, f_2) \in Z\), respectively, it is easily seen that
\[
\left\| \begin{pmatrix}
u(\cdot, t) \\
v(\cdot, t)
\end{pmatrix} - \begin{pmatrix}
\bar{\nu}(\cdot, t) \\
\bar{v}(\cdot, t)
\end{pmatrix} \right\|_H \leq \left\| \begin{pmatrix}
u_0 \\
v_0
\end{pmatrix} - \begin{pmatrix}
\bar{\nu}_0 \\
\bar{v}_0
\end{pmatrix} \right\|_H
+ \int_0^t \left\| \begin{pmatrix}
f_1(\cdot, s) \\
f_2(\cdot, s)
\end{pmatrix} - \begin{pmatrix}
\bar{f}_1(\cdot, s) \\
\bar{f}_2(\cdot, s)
\end{pmatrix} \right\|_H ds, \quad 0 \leq t \leq T.
\] (5.5)

We have used the fact that \( A(t) \) is monotone for every \( t \in [0, T] \). By (5.5) the conclusion follows. \( \square \)

It seems that such results cannot be derived from known general theorems related to time dependent evolution equations associated to monotone or accretive operators [5], [6], [11], [13]. For instance, even if \( \beta \) is linear (as in Remark 5.1), the well-known conditions of Crandall-Pazy (see [11, p. 49]) are not verified. This can easily be seen by a straightforward computation.

**5.4.** Similar higher order regularity results for our BVP can also be obtained. For instance, let us briefly point out how to proceed with the next level of regularity, i.e., \( C^2 \) regularity. Of course, in addition to (2.1) and (4.1), the data \( u_0, v_0, f_1, f_2 \) have to satisfy some second order compatibility conditions. They can be derived in a natural way. So, if \((u, v) \in C^2(\bar{Q}_T)^2 \) is a solution of BVP, then it satisfies for every \( t \in [0, T] \)
\[
\begin{align*}
\left\{ \begin{array}{l}
r_0 u_{ttt}(0, t) + v_{ttt}(0, t) = 0, \\
u_{ttt}(1, t) = \beta'(v(1, t)) \cdot v_{ttt}(1, t) + \beta''(v(1, t)) \cdot v_t^2(1, t).
\end{array} \right.
\end{align*}
\] (5.6)
On the other hand,
\[ u_{tt} = f_{1t} - ru_t - v_{xt} = f_{1t} - r(f_1 - ru - v_x) - (f_{2x} - g v_x - u_{xx}), \]
\[ v_{tt} = f_{2t} - g v_t - u_{xt} = f_{2t} - g(f_2 - g v - u_x) - (f_{1x} - ru_x - v_{xx}). \]

Therefore,
\[
\begin{aligned}
    u_{tt}(0,0) &= f_{1t}(0,0) - r[f_1(0,0) - ru_0(0) - v'_0(0)] \\
    &\quad - [f_{2x}(0,0) - g v'_0(0) - u''_0(0)] =: a_0, \\
    v_{tt}(0,0) &= f_{2t}(0,0) - g[f_2(0,0) - g v_0(0) - u'_0(0)] \\
    &\quad - [f_{1x}(0,0) - ru'_0(0) - v''_0(0)] =: b_0, \\
    u_{tt}(1,0) &= f_{1t}(1,0) - r[f_1(1,0) - ru_0(1) - v'_0(1)] \\
    &\quad - [f_{2x}(1,0) - g v'_0(1) - u''_0(1)] =: c_0, \\
    v_{tt}(1,0) &= f_{2t}(1,0) - g[f_2(1,0) - g v_0(1) - u'_0(1)] \\
    &\quad - [f_{1x}(1,0) - ru'_0(1) - v''_0(1)] =: d_0.
\end{aligned}
\]
(5.7)

Taking \( t = 0 \) in (5.6) and using (5.7) one gets the following second order compatibility conditions:
\[
\begin{aligned}
    r_0 a_0 + b_0 &= 0, \\
    c_0 &= \beta'(v_0(1))d_0 + \beta''(v_0(1)) \cdot [f_2(1,0) - g v_0(1) - u'_0(1)]^2.
\end{aligned}
\]
(5.8)

We are now in a position to formulate the following higher order regularity result:

**Theorem 5.4.** If \( \beta = \beta(\xi) \in C^2(\mathbb{R}), \beta' \geq 0; f_1, f_2 \in C^2(\bar{Q}_T); u_0, v_0 \in C^2[0,1]; \) and \( u_0, v_0, f_1, f_2 \) satisfy (2.1), (4.1), and (5.8), then BVP has a unique solution \((u,v) \in C^2(\bar{Q}_T)^2.\)

**Proof.** By Theorem 4.1, BVP has a unique solution \((u,v) \in C^1(\bar{Q}_T)^2.\) Obviously, \((u_t, v_t)\) is the unique generalized solution of class \(C(\bar{Q}_T)^2\) (see Theorem 5.1) for the following boundary value problem
\[
\begin{aligned}
    \ddot{u} + \ddot{v} + r \ddot{u} &= f_{1t}, \\
    \ddot{v} + \ddot{u} + g \ddot{u} &= f_{2t}, 0 < x < 1, 0 < t < T, \\
    \dot{r_0} \ddot{u}(0,t) + \dot{v}(0,t) &= 0, \\
    \ddot{u}(t,1) &= \beta'(v(1,t)) \cdot \ddot{v}(1,t), 0 \leq t \leq T, \\
    \ddot{u}(x,0) &= f_1(x,0) - ru_0(x) - v'_0(x), \\
    \ddot{v}(x,0) &= f_2(x,0) - g v_0(x) - u'_0(x), 0 \leq x \leq 1.
\end{aligned}
\]
(5.9)

But in fact, by Theorem 5.2 and Remark 5.1, problem (5.9)–(5.11) has a (unique) solution \((\ddot{u}, \ddot{v}) \in C^1(\bar{Q}_T)^2.\) Hence \((u_t, v_t) \in C^1(\bar{Q}_T)^2.\) This implies that \((u,v) \in C^2(\bar{Q}_T),\) because \((u,v)\) satisfies (S). \(\square\)
5.5. It seems that our results can be extended to semilinear systems, i.e., when \( ru \) and \( gv \) are replaced by nonlinear terms: \( r(x, u), g(x, v) \).

Also, it would be interesting to study the case when both boundary conditions are nonlinear, or even the case

\[
(-u(0, t), u(1, t)) \in \gamma(t, v(0, t), v(1, t)),
\]

where \( \gamma \) is a multivalued application.

REFERENCES


(Accepted December 1998)