SECOND ORDER DIFFERENCE EQUATIONS
OF MONOTONE TYPE

Gheorghe Moroșanu
Faculty of Mathematics, University of Iași, 6600 Iași, Romania

ABSTRACT

Our aim is to investigate the existence of solutions to some second order difference equations of monotone type. Theorems 1.1 and 1.2 below are the discrete versions of some existence results due to V. Barbu [1] corresponding to the continuous case.

1. INTRODUCTION AND MAIN RESULTS

Let \( H \) be a real Hilbert space with inner product \((\cdot, \cdot)\) and norm \(|\cdot|\). Consider a maximal monotone operator \( A \) from \( H \) into \( H \) with domain and range denoted by \( D(A) \) and \( R(A) \) respectively.

Let \((c_n)\) be some sequence of positive real numbers. We assume familiarity with the notations, definitions and basic facts about nonlinear monotone operators on infinite dimensional spaces. The background material in this field can be found in the books [1] and [2]. Our main results may be stated as follows:

Theorem 1.1 Let \( x \) and \( y \) be some given elements of \( H \). Then for every natural \( N \) there exists a unique \( N \)-vector \( u^N = (u^N_1, u^N_2, \ldots, u^N_N) \) belonging to \( H^N \) such that \( u_i \) belong to \( D(A) \),

Copyright © 1979 by Marcel Dekker, Inc. All Rights Reserved. Neither this work nor any part may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, microfilming, and recording, or by any information storage and retrieval system, without permission in writing from the publisher.
i = 1, 2, ..., N and $U^N$ verifies the following difference system

$$
\begin{align*}
    u_{i+1}^N &= 2u_i^N + u_{i-1}^N - c_i A^N u_i^N, & i = 1, 2, ..., N, \\
    u_0^N &= x, & u_{N+1}^N &= y.
\end{align*}
$$

Here $H^N$ denotes the product space endowed with the usual scalar product and euclidian norm.

Theorem 1.2 Suppose that $O$ belongs to $R(A)$ and let $x$ be a given element of $H$. Then there exists a unique sequence $(u_n)$ in $H$ such that $(u_n)$ is bounded and satisfies the following difference system

$$
\begin{align*}
    u_{n+1} &= 2u_n + u_{n-1} - c_n A u_n, & n = 1, 2, ... \\
    u_0 &= x
\end{align*}
$$

If in addition there exists $k_0 > 0$ such that $c_n \geq k_0$, $n = 1, 2, ...$ then there exists

$$
\lim_{n \to \infty} u_n = p, \quad \text{weakly in } H, \quad \text{where } p \quad \text{belongs to } A^{-1}O.
$$

Assuming in addition that $A$ satisfies certain coercivity conditions we obtain (1.3) in the strong topology of $H$ even if $(c_n)$ is not bounded away from zero (see Remarks 2.2 and 2.3). Under the coercivity hypotheses we shall also investigate the non-homogeneous difference equations (see Remark 2.4). If $A$ is a subdifferential then the solutions of (1.1) and (1.2) minimize certain convex functions (cf. Remark 2.6)

2. PROOFS

For simplicity we shall suppose in what follows that $A$ is single-valued.
Proof of Theorem 1.1.

We define the following two operators

\[ A^N : D(A)^N \longrightarrow H^N \text{ and } B^N : H^N \longrightarrow H^N \]

by

\[
A^N u^N = (c_1 u_1^N, c_2 u_2^N, \ldots, c_N u_N^N) + u_0^N,
\]

\[
B^N u^N = (2u_1^N - u_2^N, -u_1^N + 2u_2^N - u_3^N, -u_2^N + 2u_3^N - u_4^N, \ldots, -u_{N-2}^N + 2u_{N-1}^N - u_N^N, -u_{N-1}^N + 2u_N^N),
\]

where

\[ u^N = (u_1^N, u_2^N, \ldots, u_N^N) \text{ belongs to } H^N \]

and

\[ u_0^N = (x, 0, \ldots, 0, y) \]

It is easy to verify that \( A^N \) is maximal monotone. Since \( B^N \) is everywhere defined, linear, continuous and monotone on \( H^N \), it follows that it is maximal monotone and \( A^N + B^N \) is also maximal monotone (see [1], p.158). Taking into account the above notations it is obvious that (1.11) can be written in the following form

(2.1)

\[ A^N u^N + B^N u^N \ni 0^N, \]

where \( 0^N \) is the null element of \( H^N \). Now, we shall prove that \( B^N \) is strongly monotone, i.e. there is \( a_N > 0 \), such that

(2.2)

\[
(B^N u^N, u^N)_N \geq a_N |u^N|_N^2,
\]

for every \( u^N \) belonging to \( H^N \), where \((\cdot, \cdot)_N\) and \(|\cdot|_N\) denote the scalar product and respectively norm of \( H^N \). We have

\[ (B^N u^N, u^N)_N = 2 \sum_{i=1}^N |u_i^N|^2 - 2 \sum_{i=1}^{N-1} (u_i^N, u_{i+1}^N). \]

For every \( b_i > 0 \) we can write

\[ 2(u_i^N, u_{i+1}^N) < b_i |u_i^N|^2 + (\gamma b_i) |u_{i+1}^N|^2, \quad i = 1, 2, \ldots, N-1, \]

which implies that
\[ \sum_{i=1}^{N-1} (u_i^N, u_{i+1}^N) \leq b_1 |u_1^N|^2 + \sum_{i=1}^{N-2} (b_{i+1} + \frac{1}{b_1}) |u_{i+1}^N|^2 + \left( \frac{1}{b_{N-1}} \right) |u_N^N|^2. \]

Therefore, to obtain (2.2) suffice it to find some numbers \( b_i \) such that

\[ b_1 < 2, \quad \frac{1}{b_{N-1}} < 2, \]

\[ b_{i+1} + \frac{1}{b_1} < 2, \quad i = 1, 2, \ldots, N-2. \]

(2.3)

For example, we can choose the numbers

\[ b_i = (1+i)^2 / i(i+2), \quad i = 1, 2, \ldots, N-1 \]

to satisfy (2.3), therefore (2.2) holds. Without loss of generality we may assume that \( 0 \) belongs to \( D(A) \) which implies \( 0^N \) belongs to \( D(A^N) = D(A)^N \). Thus \( A^N + B^N \) is coercive so that \( R(A^N + B^N) = H^N \).

Therefore (2.1) has a solution. The uniqueness is a consequence of (2.2) and monotonicity of \( A^N \). This completes the proof.

**Proof of Theorem 1.2.**

Consider the following sequence of problems

\[ u_{i+1}^N = 2u_i^N + u_{i-1}^N - c_i Au_i^N, \quad i = 1, 2, \ldots, N, \]

\[ u_{N+1}^N = u_0^N = x. \]

Let \( p \) be an arbitrary element of \( F = A^{-1}p \) and denote

\[ y_i^N = u_i^N - p. \]

Multiplying (2.4) by \( y_i^N \) we obtain

\[ (y_{i+1}^N - 2y_i^N + y_{i-1}^N, y_i^N) \geq 0, \quad i = 1, 2, \ldots, N. \]

(2.5)

The last inequality yields

\[ |y_i^N| \leq (\frac{1}{2}) (|y_{i-1}^N| + |y_{i+1}^N|), \quad i = 1, 2, \ldots, N. \]

(2.6)
Since
\[ y_0^N = y_{N+1}^N = x - p \]
it is easy to show that
\[ |y_i^N| \leq |x - p|, \text{ for every } N, i = 1, 2, \ldots, N. \]

From (2.5) we derive
\[ |y_i^N - y_{i-1}^N|^2 \leq (y_{i+1}^N - y_i^N, y_i^N) - (y_i^N - y_{i-1}^N, y_{i-1}^N), \]
which yields
\[ (2.8) \quad \sum_{i=1}^{N} |u_i^N - u_{i-1}^N|^2 \leq 3|x - p|^2 \]
Let us take \( N_0 < N' < N \) and denote
\[ z_i = u_i^N - u_i^{N'}, \quad i = 0, 1, \ldots, N' + 1. \]

From (2.4) we get
\[ z_{i+1} - 2z_i + z_{i-1} \in c_1(Au_i^N - Au_i^{N'}), \quad i = 1, 2, \ldots, N'. \]

Multiplying this by \( z_i \) one obtains
\[ (2.9) \quad (z_{i+1} - 2z_i + z_{i-1}, z_i) \geq 0, \quad i = 1, 2, \ldots, N' \]
and, after some rearrangements, (2.9) leads to
\[ (2.10) \quad |z_i - z_{i-1}|^2 \leq (z_{i+1} - z_i, z_i) - (z_i - z_{i-1}, z_{i-1}), \]
\[ i = 1, 2, \ldots, N'. \]

Since
\[ |z_k|^2 = \sum_{i=1}^{k} (|z_i| - |z_{i-1}|) \leq k^{1/2} \sum_{i=1}^{k} |z_i - z_{i-1}|^2 / 2, \]
\[ k = 1, 2, \ldots, N', \]
by means of (2.10) we can write
\[ |z_k|^2 \leq k(z_{k+1} - z_k, z_k) \leq (1/2)k(|z_{k+1}|^2 - |z_k|^2) \]
and this yields
\[ (2.11) \quad \sum_{k=N_0}^{N'} (\sqrt{k}) |u_k^N - u_k^{N'}|^2 \leq (1/2)k(u_{N'+1}^N - x)^2 \leq 2|x - p|^2 \]
By (2.10) we get
\[(z_{j+1} - z_j, z_j) \equiv 0, \quad j = 1, 2, \ldots, N',\]
therefore \(|z_j|\) is increasing. This together with (2.11) implies
\[(2.12) \quad |u_i^N - u_i^{N'}| \leq 2^{1/2} \text{ dist } (x,F)(\sum_{k=N_0}^{N'} (1/k))^{-1/2},
\]
\[i = 1, 2, \ldots, N_0.\]

Thus, there exists
\[(2.13) \quad \lim_{N \to \infty} u_i^N = u_i ,\]
uniformly for \(i\) belonging to every finite set of natural numbers. Using (2.13) and closedness of \(A\) we can pass to limit in (2.4), as \(N \to \infty\) to obtain that the sequence \((u_n)\) verifies (1.2). From (2.7) it follows that \((u_n)\) is bounded. Now, we shall prove that \((u_n)\) is the unique bounded solution of (1.2). Let \((v_n)\) be some other bounded sequence satisfying (1.2). Denoting \(w_n = u_n - v_n\) we obtain
\[w_{i+1} - 2w_i + w_{i-1} \in (Au_i - Av_i), \quad i = 1, 2, \ldots,\]
which implies
\[(2.14) \quad |w_i| \leq (1/2)(|w_{i-1}| + |w_{i+1}|), \quad i = 1, 2, \ldots\]

But \((w_1)\) is bounded so that (2.14) says that \(|w_i|\) is nonincreasing. Noting that \(w_0 = 0\) we conclude that \(u_i = v_i, i = 1, 2, \ldots\). To prove the last part of Theorem 1.2 we assume in addition that \(c_n \geq k > 0\). Passing to limit in (2.6), as \(N \to \infty\) we get
\[(2.15) \quad |u_i - p| \leq (1/2)(|u_{i-1} - p| + |u_{i+1} - p|), \quad i = 1, 2, \ldots\]
where \(p\) is an arbitrary element of \(F\). Since \((u_n)\) is bounded (2.15) implies that \(|u_n - p|\) is
nonincreasing, for every $p$ in $F$, therefore there exists

$$\lim_{n \to \infty} |u_n - p| = \rho(p).$$

From (2.8) we deduce

$$\sum_{i=1}^{\infty} |u_i - u_{i-1}|^2 \leq 3(\text{dist}(x,F))^2,$$

so in particular

$$\lim_{n \to \infty} (u_n - u_{n-1}) = 0.$$  

Passing to limit in (1.2) and making use of demiclosedness of $A$ we deduce that every weak cluster point of $(u_n)$ belongs to $F$. This together with (2.16) implies (1.3) (cf. Opial's lemma, see e.g. [3], p.340) and the proof is complete. We note that in (2.4) we can take $u_{N+1}^N = y_N$ instead of $u_{N+1}^N = x$, where $(y_n)$ is a bounded sequence and the proof is similar.

**Remark 2.1**

It is easy to estimate that

$$3|u_n - p|^2 + c_n^2|Au_n|^2 \leq 3|u_{n-1} - p|^2, \quad n = 1,2,\ldots, \quad p \in F.$$

Thus

$$\sum_{i=1}^{\infty} c_i^2|Au_i|^2 \leq 3(\text{dist}(x,F))^2.$$  

In the significant case in which $A$ is the subdifferential of a lower semicontinuous proper convex function $f$ we have

$$0 \leq f(u_n) - f(p) \leq (Au_n, u_n - p),$$

where $p$ belongs to $F$ (i.e., $f(p) = \inf \{f(x), \quad x \in H\}$). If $c_n \geq k_o > 0$ by (2.18) we infer that $(f(u_n))$ converges to $\inf f$, as $n \to \infty$. The inequalities (2.17) and (2.18) provide informations
about the rate of convergence of \((u_n)\) and \((f(u_n))\) respectively.

**Remark 2.2**

Suppose that there is \(a > 0\) such that

\[
(2.19) \quad |Av - Aw| \geq |a|v - w|, \text{ for every } v, w \in D(A)
\]

and

\[
(2.20) \quad \lim_{n \to \infty} u_n = p = A^{-1}o \text{ strongly in } H.
\]

Here is the argument. From (1.2) and (2.19) we get

\[
|u_{n+1} - p|^2 + 2(u_{n+1} - p, u_{n-1} - p) + |u_{n-1} - p|^2 \geq a^2 c_n^2 |u_n - p|^2 + 4 |u_n - p|^2
\]

and since \(|u_n - p|\) is nonincreasing we find

\[
|u_n - p| \leq \prod_{i=1}^{n} (1 + a^2 c_i^2/3)^{-1/2} |x - p|.
\]

**Remark 2.3**

Suppose now that

\[
(2.21) \quad (Av - Aw, v - w) \geq a|v - w|^2, \text{ for every } v, w \in D(A) \text{ and } \sum_{i=1}^{\infty} c_i = \infty.
\]

Then (2.20) is again satisfied because we have

\[
|u_n - p| \leq \prod_{i=1}^{n} (1 + ac_i)^{-1/2} |x - p|.
\]

**Remark 2.4**

Suppose that \(c_n \leq k_o > 0, o \in A_o\) and define the following two operators

\[
A_2: l^2 \to l^2, \quad B_2: l^2 \to l^2, \text{ where } l^2 = l^2(H),
\]

\[
A_2(u_n) = (c_n u_n), \quad u_n \in D(A), \quad n = 1, 2, \ldots
\]

\[
B_2(u_n) = (-u_{n-1} + 2u_n - u_{n+1}), \text{ where } u_0 = x.
\]

We note that \(A_2\) is maximal monotone, \(B_2\) is everywhere defined, continuous and monotone on \(l^2\),
therefore $A_2 + B_2$ is maximal monotone. But $B_2$ is not coercive (see [1], p.34). Assuming that $A_2$ is coercive (for instance this is true if (2.21) holds) it follows that the non-homogeneous difference system

$$
\begin{align*}
    u_{n+1} - 2u_n + u_{n-1} &\in c_n A u_n + f_n, \ n=1,2, \ldots \\
    u_0 &= x
\end{align*}
$$

(2.22)

admits a unique solution $(u_n) \in l^2$, for every $(f_n) \in l^2$. The assumption $o \in A o$ can be replaced by $o \in R(A)$ to obtain that (2.22) admits as solution $(u_n-p) \in l^2$, $p = A^{-1} o$.

Remark 2.5

Theorems 1.1 and 1.2 guarantee only the existence of solutions to the mentioned difference schemes. In order to construct these solutions we need an adequate algorithm. For equations of the form (2.1) we indicate the algorithm given in [3]-[6] which involves the computation of the resolvent of $A^N + B^N$. From (2.12) we infer that

$$
(2.23) \quad |u_i^N - u_i| \leq \text{const.} \left( \sum_{k=N_0}^{N} \left( \frac{1}{k} \right) \right)^{-1/2}, \ i=1,2, \ldots, N_0
$$

so that the solution of (1.1) approximates the solution of (1.2). Finally, we remark that if $c_n \geq k_0 > 0$ from (1.3) and (2.23) it follows that for $N_0$ sufficiently large and $N \gg N_0$, $u^N_N$ approach the solution of $A x \in o$.

Remark 2.6

If $A$ is the subdifferential of $f$, $f$ being a lower semicontinuous proper convex function on $H$, then the solution $U^N$ of (1.1) is just the solution of the following optimization problem

Minimize $F^N(v^N) = \frac{1}{2} \sum_{i=1}^{N} |v_{i+1}^N - v_i^N|^2 + \sum_{i=1}^{N} c_i f(v_i^N)$,
for all \( v^N = (v_1^N, v_2^N, \ldots, v_N^N) \) belonging to \( H^N \) and \( v_0^N = x, v_{N+1}^N = y \). Let us assume in addition that \( f \not\equiv 0 \). Then it is easy to see that the solution \( (u_n) \) of (1.2) is the optimal solution of the following problem

\[
\text{Minimize } F(v_n) = \sum_{i=1}^{\infty} \left( \frac{1}{2} |v_i - v_{i-1}|^2 + c_i f(v_i) \right),
\]

in the class of bounded sequences \( (v_n) \) such that \( v_0 = x \). Of course, it is necessary to suppose that there exists at least one sequence \( (v_n) \) in the mentioned class such that \( F(v_n) < \infty \).

REFERENCES


