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Singularly Perturbed BVPs for the Telegraph System

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Some nonlinear boundary value problems associated with the telegraph system are discussed, in which the inductance per unit length is assumed to be a small parameter. It turns out that these problems are singularly perturbed with respect to the uniform norm. A formal first order asymptotic expansion for the solution of a particular BVP is constructed. This expansion is validated by a complete analysis, including estimates for the components of the remainder.

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1 Introduction

Let \( D_T := \{(x, t) \in \mathbb{R}^2; 0 < x < 1, 0 < t < T\} \), where \( T > 0 \) is a given time instant. Consider in \( D_T \) the telegraph system

\[
\begin{align*}
\varepsilon u_t + v_x + ru &= f_1(x, t), \\
\alpha v_t + u_x + gv &= f_2(x, t),
\end{align*}
\]

(\(LS\))

with initial conditions

\[
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad 0 \leq x \leq 1,
\]

(\(IC\))

and nonlinear boundary conditions of the form

\[
\begin{align*}
u(0, t) + c_0v_t(0, t) &= -h_0(v(0, t)) - t_0 \int_0^t v(0, s) ds + e_0(t), \\
u(1, t) - c_1v_t(1, t) &= h_1(v(1, t)) + t_1 \int_0^t v(1, s) ds - e_1(t), \quad 0 \leq t \leq T.
\end{align*}
\]

(\(BC\))

Problem \((LS), (IC), (BC)\) is a model for transmission (propagation) in electrical circuits. Here, \(u(x, t)\) represents the current at time instant \(t\), at point \(x\) of the line; \(v(x, t)\) is the voltage at \(t\) and \(x\); \(r, g, \varepsilon, c\) represent the resistance, conductance, inductance and capacitance per unit length; \(f_1\) represents the voltage per unit length impressed along the line in series with it. In practice, \(f_2 = 0\), and we assume without any loss of generality that \(c = 1\). Inductance \(\varepsilon\) is assumed to be a positive small parameter. It is well known that the inductance of the line is small whenever the corresponding frequency is small. For the physical interpretation of \((BC)\) we refer the reader to [1], [2]. In general, these are dynamic boundary conditions due to the presence of \(v_t(0, t)\), \(v_t(1, t)\). In this paper, we restrict our attention to the following particular case of boundary conditions:

\[
\begin{align*}
r_0u(0, t) + v(0, t) &= 0, \\
u(1, t) - c_1v_t(1, t) &= h_1(v(1, t)) + e_1(t), \quad 0 \leq t \leq T.
\end{align*}
\]

(\(BC:1\))

Note that the above boundary condition at \(x=0\) represents the usual Ohm’s law. At the other end of the line we have a nonlinear boundary condition (due to a nonlinear resistor). Here, \(r_0, c_1\) are positive constants, \(h_1\) is a continuous nondecreasing function, and \(e_1\) is a smooth function representing the source at \(x = 1\).

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We are going to display some results which will be included into [1]. First of all, one can show that problem \((LS), (IC), (BC.1)\), also denoted \(P_\varepsilon\), is singularly perturbed with respect to the uniform convergence in \(\overline{D_T}\), with a boundary layer located near the segment \(\{(x,0); 0 \leq x \leq 1\}\).

2 Asymptotic expansion

In this section we derive formally a first order asymptotic expansion for the solution \(U_\varepsilon = (u_\varepsilon(x,t), v_\varepsilon(x,t))\) of problem \(P_\varepsilon\). More precisely, we seek \(U_\varepsilon\) in the form:

\[
U_\varepsilon = U_0(x,t) + \varepsilon U_1(x,t) + V_0(x,\tau) + \varepsilon V_1(x,\tau) + R_\varepsilon(x,t), \quad (x,t) \in D_T,
\]

where:
- \(\tau = t/\varepsilon\) is the fast (rapid) variable;
- \(U_0 = (X_0(x,t), Y_0(x,t)), U_1 = (X_1(x,t), Y_1(x,t))\) are the first two terms of the corresponding regular series;
- \(V_0 = (j_0(x,\tau), k_0(x,\tau)), V_1 = (j_1(x,\tau), k_1(x,\tau))\) are boundary layer functions;
- \(R_\varepsilon = (R_\varepsilon(x,t), R_\varepsilon(x,t))\) is the remainder of order one.

We require that \(U_\varepsilon\) given by (1) satisfy formally \(P_\varepsilon\) and identify the coefficients of the like powers of \(\varepsilon^k, k = -1,0,1\).

For the components of the zeroth order regular term we obtain

\[
\begin{align*}
X_0 &= (1/r)(f_1 - Y_{0x}), \\
Y_{0t} - (1/r)Y_{0xx} + gY_0 &= f_2 - (1/r)f_{1x} \text{ in } D_T, \\
Y_0(x,0) &= u_0(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

(2)

(3)

For the zeroth order boundary layer functions we obtain the formulae

\[
j_0(x,\tau) = \alpha(x)e^{-\tau}, \quad k_0(x,\tau) \equiv 0,
\]

(4)

where function \(\alpha\) has the following expression which is derived from \((IC)\),

\[
\alpha(x) = u_0(x) + (1/r)(v'_0(x) - f_1(x,0)).
\]

For the components of the first order regular term we can easily derive the system

\[
\begin{align*}
X_{0t} + Y_{1x} + rX_1 &= 0, \\
Y_{1t} + X_{1x} + gY_1 &= 0 \text{ in } D_T,
\end{align*}
\]

(5)

which can be written in the following equivalent form

\[
\begin{align*}
X_1 &= -(1/r)(X_{0t} + Y_{1x}), \\
Y_{1t} - (1/r)Y_{1xx} + gY_1 &= (1/r)X_{0xt} \text{ in } D_T.
\end{align*}
\]

(6)

(7)

The components of the first order remainder satisfy formally

\[
\begin{align*}
\varepsilon R_{1xt} + R_{2xx} + rR_1 &= -\varepsilon^2 X_{1t}, \\
R_{2xt} + R_{1xx} + gR_2 &= -\varepsilon(j_{1x} + gk_1) \text{ in } D_T.
\end{align*}
\]

(8)

The first order boundary layer functions satisfy the system

\[
\begin{align*}
j_{1t} + k_{1x} + rj_1 &= 0, \\
k_{1t} + j_{0x} &= 0,
\end{align*}
\]

and so it is easily seen that

\[
\begin{align*}
j_1(x,\tau) &= \beta(x)e^{-\tau} - (\alpha''(x)/r)e^{-\tau}, \\
k_1(x,\tau) &= (1/r)\alpha'(x)e^{-\tau},
\end{align*}
\]
where function $\beta$ can be determined from (IC). Indeed, from (IC) we get
\[ X_1(x,0) + j_1(x,0) = 0 \iff \]
\[ \beta(x) = (1/r)X_{0t}(x,0) - (1/r^2)\alpha''(x), \quad 0 \leq x \leq 1, \quad (9) \]
\[ k_1(x,0) + Y_1(x,0) = 0 \iff \]
\[ Y_1(x,0) = -(1/r)\alpha'(x), \quad 0 \leq x \leq 1. \quad (10) \]
For the remainder components we obtain the initial conditions:
\[ R_{1x}(x,0) = R_{2x}(x,0) = 0, \quad 0 \leq x \leq 1. \quad (11) \]
Now, let us use (BC.1). We first derive the equation
\[ r_0X_0(0,t) + Y_0(0,t) = 0, \quad 0 \leq t \leq T, \]
which yields (see (2)):
\[ Y_0(0,t) = -(r_0/r)Y_{0x}(0,t) = -(r_0/r)f_1(0,t), \quad 0 \leq t \leq T. \quad (12) \]
Next, we derive the condition $j_0(0,\tau) = 0$, i.e.,
\[ \alpha(0) = 0 \iff r\alpha(0) = f_1(0,0) - \nu_0(0). \quad (13) \]
We can also derive from (BC.1):
\[ r_0X_1(0,t) + Y_1(0,t) = 0, \quad 0 \leq t \leq T, \]
and, consequently (see (6)),
\[ Y_1(0,t) = -(r_0/r)Y_{1x}(0,t) = (r_0/r)X_{0t}(0,t), \quad 0 \leq t \leq T. \quad (14) \]
In addition, we obtain the equation
\[ r_j1(0,\tau) + k_1(0,\tau) = 0, \]
i.e.,
\[ \begin{align*}
\alpha''(0) &= 0, \\
\alpha'(0) + r\alpha(0) &= 0. 
\end{align*} \quad (15) \]
From the latter boundary condition we get
\[ X_0(1,t) - c_1Y_{0t}(1,t) = h_1(Y_0(1,t)) + e_1(t), \quad 0 \leq t \leq T, \]
\[ X_1(1,t) - c_1Y_{1t}(1,t) = h_1(Y_0(1,t))Y_1(1,t), \quad 0 \leq t \leq T. \]
Obviously, these equations can be rewritten (see (2), (6)) as:
\[ c_1Y_{0t}(1,t) + (1/r)Y_{0x}(1,t) + h_1(Y_0(1,t)) = (1/r)f_1(1,t) - e_1(t), \quad (16) \]
\[ c_1Y_{1t}(1,t) + (1/r)Y_{1x}(1,t) + h_1(Y_0(1,t))Y_1(1,t) = -(1/r)X_{0t}(1,t), \quad (17) \]
for all $t \in [0,T]$. Another equation given by the identification procedure is
\[ j_0(1,\tau) - c_1k_{1x}(1,\tau) = 0, \]
which is satisfied if and only if
\[ \alpha(1) + c_1\alpha'(1) = 0. \quad (18) \]
Note that equations (13), (15) and (18) guarantee that our boundary layer functions \( j_0, j_1 \) and \( k_1 \) do not introduce any discrepancies at the corner points \((0, 0)\) and \((1, 0)\) of \(D_T\). These equations are also among the compatibility conditions which we need to require to get enough smoothness for the terms of expansion (1). Smoothness is also necessary for proving estimates for the components of the remainder.

Let us also point out that, unlike \( R_0, k_1 \) is not zero. This means that the boundary layer phenomenon comes into effect for the latter component of the solution starting with the second term (first order term) of the asymptotic expansion.

Finally, it is easily seen that \( R_\varepsilon \) should satisfy the following boundary conditions:

\[
\begin{align*}
    r_0 R_{1e}(0,t) + R_{2e}(0,t) &= 0, \\
    R_{1e}(1,t) - c_1\varepsilon R_{2e}(1,t) &= h_1(v_e(1,t)) - h_1(Y_0(1,t)) \\
    -\varepsilon f h_1'(Y_0(1,t))Y_1(1,t) - \varepsilon j_1(1,\tau) &= 0 < t < T.
\end{align*}
\]

Note that the first term of expansion (1), \( U_0 = (X_0(x,t), Y_0(x,t)) \), satisfies the reduced problem \( P_0 \) which is made up by the algebraic equation (2) and the boundary value problem (2)_2, (3), (12), (16).

In conclusion, if there are smooth solutions for \( P_\varepsilon, \varepsilon > 0, P_0 \), and for problem (6), (7), (10), (14), (17), denoted by \( P_1 \), then \( U_\varepsilon \) can be represented by (1), where \( R_\varepsilon = (R_{1e}, R_{2e}) \) satisfies (8), (11), (19).

### 3 Validation of our expansion and estimates for the remainder components

In order to show that expansion (1) is well defined, we need to prove that problems \( P_\varepsilon, P_0, P_1 \) have solutions which are sufficiently smooth. Since these problems include dynamic BCs, the existence of solutions should be discussed within appropriate spaces. To derive higher regularity, we have to handle time-dependent evolution equations. Obviously, smoothness of the solutions requires adequate regularity conditions for the data as well as enough compatibility with the BCs. Smoothness of the solutions is also necessary in deriving estimates for the components of the remainder. In particular, we have:

**Theorem 3.1** Assume that all the data are smooth enough and satisfy adequate compatibility conditions. Then, for every \( \varepsilon > 0 \), the solution of problem \( P_\varepsilon \) admits an asymptotic expansion of the form (1) and the following estimates hold:

\[
\begin{align*}
    \|R_{1e}\|_{C(D_T)} &= O(\varepsilon^{9/8}), \quad \|R_{2e}\|_{C(D_T)} = O(\varepsilon^{11/8}), \\
    \|R_{1e}\|_{C([0,T];L^2(0,1))} &= O(\varepsilon^{1/2}), \quad \|R_{2e}\|_{C([0,T];L^2(0,1))} = O(\varepsilon), \\
    \|R_{1e}\|_{C([0,T];L^2(0,1))} &= O(\varepsilon^{5/4}), \quad \|R_{2e}\|_{C([0,T];L^2(0,1))} = O(\varepsilon^{5/4}).
\end{align*}
\]

Since the remainder is \( O(\varepsilon^\alpha) \), with \( \alpha > 1 \), it follows that (1) is a real first order expansion.

In [1] we examine various boundary conditions for the telegraph equations, as well as other singularly perturbed BVPs, especially nonlinear problems which have hardly been studied. Their asymptotic analysis is very interesting but requires specific methods and tools.

### References
