TIME PERIODIC SOLUTIONS FOR A CLASS OF HYPERBOLIC PARABOLIC DIFFERENTIAL SYSTEMS

G. MOROŞANU and D. PETROVANU

The aim of this paper is to investigate the existence of solution to the following periodic problem

\[ \begin{align*}
\frac{\partial u}{\partial t} + T_x u + A(x, u) &= f(t, x) \\
\frac{\partial v}{\partial t} - T_x u + B(x, u) &= g(t, x)
\end{align*} \]

\((0 < x < 1, \ 0 < t < \omega)\)

\[ \begin{pmatrix}
(L^*_1 u)(t, 0) \\
(L^*_1 u)(t, 1)
\end{pmatrix} = \begin{pmatrix}
\int_{\omega}^t v(t, 0)
\int_{\omega}^t v(t, 1)
\end{pmatrix}
\]

\((BC)\)

\[ \begin{pmatrix}
(L^*_n u)(t, 0) \\
(L^*_n u)(t, 1)
\end{pmatrix} = \begin{pmatrix}
\frac{\partial^{n-1} v}{\partial x^{n-1}}(t, 0) \\
\frac{\partial^{n-1} v}{\partial x^{n-1}}(t, 1)
\end{pmatrix} \quad (0 < t < \omega).
\]

\((PC)\)

\[ u(0, x) = u(\omega, x), \quad v(0, x) = v(\omega, x) \quad (0 < x < 1), \]

where \(T_x u := \sum_{k=0}^{n} a_k(x) \frac{\partial^k u}{\partial x^k}, \) \(T_x^* u := \sum_{k=0}^{n} (-1)^{k} \frac{\partial^k}{\partial x^k} [a_k(x) u],\) while \(L_j (j = 1, 2, ..., n)\) are defined by

\[ L_j u := \sum_{k=0}^{n} (-1)^{j+k} \frac{\partial^k}{\partial x^k} a_j(x) u. \]

The mapping \(l\) in \((BC)\) is assumed to be maximal monotone in the space \(R^n\) and possibly multi-valued. So the symbol \(\equiv\) (a member of) in \((BC)\) makes sense.

We recall that the problem made up by \((S), (BC)\) and the initial conditions

\[ u(0, x) = u(\omega, x), \quad v(0, x) = v(\omega, x) \quad (0 < x < 1), \]

where \(T_x u := \sum_{k=0}^{n} a_k(x) \frac{\partial^k u}{\partial x^k}, \) \(T_x^* u := \sum_{k=0}^{n} (-1)^{k} \frac{\partial^k}{\partial x^k} [a_k(x) u],\) while \(L_j (j = 1, 2, ..., n)\) are defined by

\[ L_j u := \sum_{k=0}^{n} (-1)^{j+k} \frac{\partial^k}{\partial x^k} a_j(x) u. \]
has been studied in a previous paper of the authors [5]. Notice also that the special case $n = 1$ of problem (S), (BC), (IC) has been extensively studied in the last years. References concerning this problem and its applications in hydrodynamics and electrical network theory can be found in the recent book of the first author [4]. The case $n = 2$ seems to be of interest in elastic beam theory. It is also important to emphasize that many classical boundary conditions can be obtained by making suitable choices of $l$ and $a_k$ (such that they satisfy the subsequent assumptions). For example, if $l$ is the subdifferential of the indicator function of a singleton, say \{col$(p_0, q_0, \ldots, p_{n-1}, q_{n-1})$\}, then (BC) can be written as

$$
(\partial^j v / \partial x^j)(t, 0) = p_j, \quad (\partial^j v / \partial x^j)(t, 1) = q_j \quad (j = 0, 1, \ldots, n - 1)
$$

(*bilocal boundary conditions*).

Another particular case we can obtain is that of *space periodic boundary conditions*. Indeed, if $a_k$ are constant functions, $a_n \neq 0$, and we take $l$ to be the subdifferential of the indicator function of the set \{col$(r_0, s_1, \ldots, r_{n-1}, s_n) \in \mathbb{R}^n / r_i = s_i \quad (j = 1, \ldots, n)$\} then (BC) becomes

$$
(\partial^j u / \partial x^j)(t, 0) = (\partial^j u / \partial x^j)(t, 1) \quad (j = 0, 1, \ldots, n - 1)
$$

$$
(\partial^j v / \partial x^j)(t, 0) = (\partial^j v / \partial x^j)(t, 1) \quad (j = 0, 1, \ldots, n - 1)
$$

Now, let us state the assumptions we shall use in the following.

(H.1) $l : D(l) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a maximal monotone mapping, possibly multivalued.

(H.2) $a_k \in W^k, \infty(0, 1)$ $(k = 0, 1, \ldots, n)$ and $a_n \neq 0$ in $[0, 1]$.

(H.3) The functions $x \rightarrow A(x, \xi)$, $x \rightarrow B(x, \xi)$ are in $L^\infty(0, 1)$, for each $\xi \in \mathbb{R}$ and, besides, the functions $\xi \rightarrow A(x, \xi)$, $\xi \rightarrow B(x, \xi)$ are continuous and nondecreasing from $\mathbb{R}$ into $\mathbb{R}$, for a.e. $x \in [0, 1]$.

(H.4) There exist constants $a > 0$, $b > 0$ and functions $c, d \in L^2(0, 1) \cap \mathbb{R}$ such that

\[
A(x, \xi) \geq a |\xi|^2 - c(x), \quad B(x, \xi) \geq b |\xi|^2 - d(x), \quad (x \in \mathbb{R}, \text{ a.e. } x \in [0, 1]).
\]

(H.5) $f, g \in L_\text{loc}^1(R ; L^2(0, 1))$ and both $f$ and $g$ are $\omega$-periodic in time, i.e.

\[
f(l, x) = f(l + \omega, x), \quad g(l, x) = g(l + \omega, x) \quad \text{a.e.} \quad (l, x) \in \mathbb{R} \times [0, 1].
\]

In [5] we have proved that under assumptions (H.1) — (H.3) the operator $Q : D(Q) \subset X \rightarrow X$, $X = L^2(0, 1) \times L^2(0, 1)$, defined as below is maximal monotone.

(1) $D(Q) = \{(u, v) \in H^m(0, 1) \times H^m(0, 1) \mid u = u(x) \text{ and } v = v(x) \text{ satisfy } (BC)\}$.

(2) $Q \left[ \begin{array}{c} u \\ v \end{array} \right] = \left[ \begin{array}{c} T^* u \\ -T^* v \end{array} \right] + \left[ \begin{array}{c} A(\cdot, \cdot) \\ B(\cdot, \cdot) \end{array} \right]$

where $T^* = \sum_{k=0}^{\infty} a_k b(k)$ and $T^*$ is the formally adjoint of $T$. By $H^m(0, 1)$ we
have denoted the usual Sobolev space. Using essentially the fact that $Q$ is maximal monotone we have proved in the quoted paper the existence of strong and weak solutions (it depends on the choice of $u_0$, $v_0$, $f$ and $g$) to the following Cauchy problem

\[
\frac{d}{dt} \left[ \begin{array}{c}
u \\ v 
\end{array} \right] + Q \left[ \begin{array}{c}
u \\ v 
\end{array} \right] = \left[ \begin{array}{c} f(t, \cdot) \\ g(t, \cdot) 
\end{array} \right], \text{ in } X = L^2(0, 1) \times L^2(0, 1).
\]

Now, as $Q$ is a maximal monotone operator, we have by a result of [5], we have that $Q$ is dense in $X$. A strong (weak) solution of equation (E) is called a strong (respectively weak) solution of (S), (BC). For example, if $u_0, v_0 \in L^2(0, 1)$ and $f, g \in L^1_{\text{loc}}(\mathbb{R}^+; L^2(0, 1))$ then (S), (BC), (IC) has a unique weak solution $(u, v) \in C(\mathbb{R}^+; X)$. It is sufficient to recall that $D(Q)$ is dense in $X$ (see [5]).

Now, we state and prove the first result of this paper.

**Lemma 1.** If (H.1) - (H.4) hold, then $Q$ defined by (1) and (2) is coercive with respect to any $(\bar{u}_0, \bar{v}_0) \in D(Q)$, i.e.

\[
\lim_{\|u\|_X \to \infty} \frac{\langle Q \left[ \begin{array}{c} u \\ v 
\end{array} \right], \left[ \begin{array}{c} u \\ v 
\end{array} \right] - \left[ \begin{array}{c} \bar{u}_0 \\ \bar{v}_0 
\end{array} \right] \rangle_X}{\| u \|_X} = +\infty.
\]

**Proof.** Let $(\bar{u}_0, \bar{v}_0)$ be arbitrary but fixed in $D(Q)$. Then, by virtue of (H.3), (H.4), we have

\[
\langle Q \left[ \begin{array}{c} u \\ v 
\end{array} \right], \left[ \begin{array}{c} u \\ v 
\end{array} \right] - \left[ \begin{array}{c} \bar{u}_0 \\ \bar{v}_0 
\end{array} \right] \rangle_X \geq \langle \left[ \begin{array}{c} T \bar{v}_0 \\ -T^* \bar{u}_0 
\end{array} \right], \left[ \begin{array}{c} u \\ v 
\end{array} \right] - \left[ \begin{array}{c} \bar{u}_0 \\ \bar{v}_0 
\end{array} \right] \rangle_X + \int_0^1 A(x, u)(u - \bar{u}_0) \, dx + \int_0^1 B(v, x)(v - \bar{v}_0) \, dx.
\]

Example 2. Let $A$ satisfy (H.4). Assume $A$ is also uniformly elliptic. It holds that $A$ is uniformly elliptic variable and that there exists a $c > 0$ such that

\[
\lim_{\|u\|_X \to \infty} \langle u, \bar{u}_0 \rangle = R(A^{1/2}(x, u), \bar{u}_0) = +\infty.
\]

Then $A$ satisfies (H.4) as well. Indeed, by using (H.1) and the Laplace theorem, we have

\[
\lim_{\|v\|_X \to \infty} \langle v, \bar{v}_0 \rangle = R(A^{1/2}(x, v), \bar{v}_0) = +\infty.
\]

Hence $A$ satisfies (H.4) with $c(x, v)$.

where \( \| \cdot \| \) denotes the norm of $L^2(0, 1)$. Here is the argument. As $A$ is non-decreasing with respect to the second variable, we can write
\[ A(x, u)(u - \tilde{u}_0) = |A(x, u) - A(x, \tilde{u}_0)| \cdot |u - \tilde{u}_0| + A(x, \tilde{u}_0)(u - \tilde{u}_0). \]

Hence

\[ A(x, u)(u - \tilde{u}_0) \geq |A(x, u)| \cdot |u - \tilde{u}_0| - 2|A(x, \tilde{u}_0)| \cdot |u - \tilde{u}_0|. \]

Now, hypothesis (H.4) comes into play, thus obtaining

\[ A(x, u)(u - \tilde{u}_0) \geq a|u| \cdot |u - \tilde{u}_0| - (c(x) + 2|A(x, u_0)|) \cdot |u - \tilde{u}_0|. \]

Therefore,

\[ A(x, u)(u - \tilde{u}_0) \geq au^2 - a|u| \cdot |\tilde{u}_0| - c(x)|u| - c(x)|\tilde{u}_0| - 2|A(x, \tilde{u}_0)| \cdot |u - \tilde{u}_0|, \]

and this implies that

\[ \begin{cases} A(x, u)(u - \tilde{u}_0) \geq au^2 - a|u| \cdot |\tilde{u}_0| - c(x)|u| - c(x)|\tilde{u}_0| - 2|A(x, \tilde{u}_0)| \cdot |u - \tilde{u}_0|, \\ \end{cases} \]

where

\[ C_1 = a||\tilde{u}_0||_x + |c|_x + 2|A(\cdot, \tilde{u}_0)|_x, \]

\[ C_2 = (|c|_x + 2|A(\cdot, \tilde{u}_0)|_x) \cdot ||\tilde{u}_0||_x. \]

But (6) implies (4) and (5) follows by repeating word for word the same argument as above. Q.E.D.

Now, we are in a position to prove the main result of this paper.

**Theorem 1.** If (H.1) - (H.5) hold, then eq. (E) (hence problem (S), (BC)) has at least one time periodic (weak) solution with period \( \omega \).

**Proof.** Let us fix \((\tilde{u}_0, \tilde{v}_0) \in D(\tilde{Q})\). As \( \tilde{Q} \) is coercive with respect to \((\tilde{u}_0, \tilde{v}_0)\) (see Lemma 1) we can see that \( \tilde{Q} \) defined by

\[ D(\tilde{Q}) = \{(u, v) \in H^\infty((0, 1) \times H^\infty((0, 1)) | (u + \tilde{u}_0, v + \tilde{v}_0) \in D(Q)\}, \]

\[ \tilde{Q} = \tilde{Q} \left( \begin{bmatrix} u \\ v \end{bmatrix} \right) = Q \left( \begin{bmatrix} u \\ v \end{bmatrix} \right) + \tilde{Q} \left( \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \]

is coercive in the usual sense, i.e.

\[ \lim_{\|u, v\|_x \to \infty} \frac{\langle \tilde{Q} \left( \begin{bmatrix} u \\ v \end{bmatrix} \right), \left( \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \rangle}{\|u, v\|_x} = +\infty. \]

On the other hand, by making the change of functions

\[ \hat{u}(t, x) = u(t, x) - \tilde{u}_0(x), \]

\[ v(t, x) = v(t, x) - \tilde{v}_0(x), \]

problem (E), (PC) becomes.
Now, as \( \bar{Q} \) is maximal monotone and coercive we have by a result of A. Haraux [3, p. 207] that all the solutions of eq. (E) (hence of (E)) are bounded on the positive half-axis:

\[
\sup_{t \geq 0} \left( \|u(t, \cdot )\|_\infty + \|v(t, \cdot )\|_\infty \right) < +\infty.
\]

Then, we can see that problem (E), (PC) has at least one solution. To this end we can use a fixed point theorem due to F.E. Browder and W.V. Petryshyn [2]. Indeed, this theorem can be applied to the operator \( O : D(\bar{Q}) \to D(\bar{Q}) \) which associates to each initial value \( (u_0, v_0) \) the value at \( t = \omega \) of the corresponding solution of problem (E), (IC). It is easy to see that \( O \) is non-expansive and \( \{ O^n [u_0] \}_{n=1}^\infty \) is bounded in \( X \), i.e. the assumptions of the quoted Browder-Petryshyn theorem are fulfilled. Hence \( O \) has at least one fixed point. (This argument has also been used by A. Haraux [3] in a different context).

**Examples of functions satisfying (H.3) and (H.4).**

**Example 1.** The Torricelli nonlinearities

\[
A(x, \xi) = C \xi |\xi|^{z-1} \quad (C > 0, \ z \geq 1)
\]

satisfy both assumptions (H.3) and (H.4). Indeed (H.3) is trivially satisfied while (H.4) is an immediate consequence of the well-known inequality

\[
(\forall) \ r_1, r_2 \geq 0, \ p \geq 1, \ r_1, r_2 \leq r_1^p/p + r_2^q/q \ (1/p + 1/q = 1).
\]

(If \( \alpha = 1 \), then (H.4) is trivially satisfied and so in this case we do not need ineq. (8)).

**Example 2.** Let \( A \) satisfy (H.3). Assume in addition that \( A \) is everywhere differentiable on \( \mathbb{R} \) with respect to the second variable and that there exists \( a > 0 \) such that

\[
(\partial A/\partial \xi)(x, \xi) \geq a \quad \text{for a.e. } x \in [0, 1] \text{ and } \xi \in \mathbb{R}.
\]

Then \( A \) satisfies (H.4) as well. Indeed, by using (9), (H.3) and the Lagrange theorem, we have

\[
|A(x, \xi)| \geq |A(x, \xi) - A(x, 0)| - |A(x, 0)| \geq a \xi - |A(x, 0)|.
\]

Hence \( A \) satisfies (H.4) with \( c(x) = |A(x, 0)| \).

Let us remark that Torricelli's nonlinearities do not belong to the class considered in this example (because estimate (9) is violated).
Remark. If $Q$ is strongly monotone (this fact is true if, for instance, $A(x,.)$ and $B(x,.)$ are strongly monotone), then problem (E), (PC) (hence (S), (BC), (PC)) has a unique strong $\omega$-periodic solution, whenever $f, g \in W^{1,1}(0, \omega; L^{1}(0,1))$ with $f, g$ $\omega$-periodic in time (Cf. V. Barbu [1, p. 138]).

REFERENCES


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Department of Mathematics
University of Iași
Iași 7000 Iași România