WELL-POSED NONLINEAR PROBLEMS IN THE THEORY OF
ELECTRICAL NETWORKS WITH DISTRIBUTED AND LUMPED PARAMETERS

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1. Introduction

Integrated circuit modelling has not been done under a unified approach: while, strictly speaking, all integrated circuit elements should be modelled in terms of distributed parameters, the difficulties in operating with such models determine the engineering practice to use either global or partial approximation in terms of lumped parameters, depending on the problem under study.

Let us consider, for instance, the MOS-inverter illustrated in Fig. 1, a basic cell used in digital integrated circuits.

![Mos Inverter Circuit Diagram](image)

**Fig. 1. A MOS-inverter example**

When studying the effects of the interconnection line upon the circuit performance (among which the delay-time is of a primary concern), a natural "exact" model of the line $L$ is obviously represented by telegraph equations:

\[
\begin{aligned}
- \frac{\partial v_1}{\partial x} &= r_1(x)i_1(t, x) \\
- c_1(x) \frac{\partial v_1}{\partial t} &= - \frac{\partial i_1}{\partial x} - g_1(x, v_1(t, x)).
\end{aligned}
\]

(1)

In the above, $i_1(t, x)$ and $v_1(t, x)$ are the current and voltage, respectively, at the point $x \in (0, d)$ on the line at the time $t \geq 0$, $r_1(x)$ is the (unit length) resistance of the conducting layer (either polysilicon or metal) and $c_1(x)$ is the (unit length) capacitance of the oxide substrate. Any variation in the layer thickness, which is technologically unavoidable, results in the dependence of these parameters on the $x$-coordinate. The term $g_1(x, v_1)$ represents the dielectric leakage current, depending on the $x$-coordinate as well as the voltage level. The importance of modelling dielectric loss and leakage was long ago remarked in the literature [1–6], but according to our knowledge $g_1(x, v_1)$ has never been considered nonlinear with the voltage as below.

The rest of the circuit elements are modelled in terms of lumped parameters (see Fig. 2).

![Equivalent Circuit Diagram](image)

**Fig. 2. A possible equivalent network for fig. 1**

The pull-down transistor $T_1$ is considered, according to most studies referring to the delay [7], [8], as a perfect contact in its initial state (when the input voltage is positive) and as an open-circuit instantaneously realized when the input voltage decreases under zero level.

The transistor $T_2$ is always saturated and its treatment as a nonlinear resistor is essential in the study of the delay time [8]. The drain current may be described, as function of the drain-source voltage, by [9]:

\[
i = f(u) = \begin{cases}
0 & \text{when } u \leq V_T \\
K(u - V_T)^2 & \text{when } u > V_T
\end{cases}
\]

(2)

or, by using an "idealized" model with an infinite-value slope for $u > V_T$, as a multi-valued function.
\[ i \in f(u) = \begin{cases} 
0 & \text{when } u < V_T \\
[0, \infty) & \text{when } u = V_T \\
\emptyset & \text{when } u > V_T. 
\end{cases} \]  

(3)

In the above \( V_T \) is the threshold voltage, \( K \) is a positive constant and \( \emptyset \) is the void set. Both types of above nonlinearities are presented in Fig. 3.

\[ \gamma_0 \nu^a \text{diag} \{ s_1, \ldots, s_m \} \]

Fig. 3. Nonlinear characteristics

The inverter load is taken into consideration by the lumped elements \( R_1, R_2, s, \) such that we obtain:

\[
\begin{align*}
\left\{ 
& i_1(t,0) \in f(V-v_1(t,0)) \\
& i_1(t,d) = \frac{[v_1(t,d) - v_2(t)]}{R_1} \\
& i_2(t) = s \frac{dv_2}{dt} = \frac{[v_1(t,d) - v_2(t)]}{R_1} - \frac{v_2(t)}{R_2},
\end{align*}
\]  

(4)

where the symbol \( \in \) substitutes the \( = \) sign when taking into account \( f \) as a multivalued function. We note also that \( V \) is the direct current bias source.

The model including equations (1) and (4) is to be completed by the initial conditions. When taking transistor \( T_1 \) as a perfect switch, the initial conditions are not consistent with the boundary conditions of (4), [10].

While remembering this model, we state a general problem where (see Fig. 4), a resistive nonlinear 2n+m-port with time constant sources is connected between \( m \) capacitors and \( n \) distributed parameters elements.

These last ones are described by (see (1)):

\[
\begin{align*}
& \left\{ 
& \frac{c_k(x)}{\partial x} \frac{\partial v_k}{\partial t} = \partial x \left( \frac{1}{r_k(x)} \frac{\partial v_k}{\partial x} \right) - g_k(x, v_k(t,x)),
\end{align*}
\]  

(E.)

\[ k = 1, \ldots, n, x \in (0,d_k), \ t > 0. \]

Let us denote

\[
\begin{align*}
\nu^a &= \text{col} \{ v_1, \ldots, v_n \}, \\
\nu^b &= \text{col} \{ v_{n+1}, \ldots, v_{n+m} \}, \\
\gamma_0 \nu^a &= \text{col} \{ v_1(t,0), v_1(t,d) , \ldots, v_n(t,0), v_n(t,d) \}, \\
\gamma_1 \nu^a &= \text{col} \{ \frac{1}{\partial x} \frac{\partial v_1}{\partial x}(t,0) \frac{\partial v_1}{\partial x}(t,d), \ldots, \frac{1}{\partial x} \frac{\partial v_n}{\partial x}(t,0) \frac{\partial v_n}{\partial x}(t,d) \}, \\
v &= \begin{bmatrix} v^a \ v^b \end{bmatrix} = (v^a, v^b), \\
S &= \text{diag} \{ s_1, \ldots, s_m \}.
\end{align*}
\]

Taking into account the multiport equation \( -j \in \mathbb{G}(w) \) and relations (see again Fig. 4), \( j_{2k} = j_{2k-1} = i_k(t,0), \ j_{2k} = -i_k(t,d_k), w_{2k-1} = v_k(t,0), w_{2k} = v_k(t,d_k) \) for \( k = 1, \ldots, n \) and \( j_{2n+m} = i_{2n+m}(t), w_{2n+m} = v_{n+k}(t) \) for \( k = 1, \ldots, m, \) we obtain:

\[
\begin{bmatrix} - \left[ \frac{\gamma_1 \nu^a}{\partial x} \right] \end{bmatrix} \in \mathbb{G} \begin{bmatrix} \gamma_0 \nu^a \\
\nu^b \end{bmatrix}.
\]  

(B.C.)

Fig. 4. The network under study

It is natural to complete the above with initial conditions

\[
\begin{align*}
\nu^a(0,x) &= \text{col} \{ v_{10}(x), \ldots, v_{u0}(x) \} \equiv v_0^a(x), \\
\nu^b(0) &= \text{col} \{ v_{n+1,0}, \ldots, v_{n+m,0} \} \equiv v_0^b.
\end{align*}
\]  

(I.C.)

The goal of the present paper is to study the consistency of the model (E.)+(B.C.)+(I.C.). We thus formulate existence and uniqueness conditions for a solution (in a well-stated sense) for both the dynamic and steady-state regime. The asymptotic behaviour is also considered.

The paper is extending the studies [10–13] through the following new points of view:

- the distributed parameter elements exhibit non–homogeneous \( r \) and \( c \) parameters
- the dielectric leakages are taken as non-linear
- the interconnecting multiport is non-linear and multivalued.
On the other hand the paper extends some results from [14], [15], where (B.C.) do not contain differential equations.

The proofs use the monotone operators theory in Hilbert spaces [18], [15] and are given elsewhere. Although the notations and the terms are standard (those from [15] or [18]), we recall them in the next section to facilitate the reading of the results given in Section 3.

Other recent studies of the correctness of distributed parameter models for electrical structures are [6], [16], [17]. As regard the delay time problem the reader can see the references quoted in [12].

2. Notations

We denote by $R^n_+$ the space $R^m$ with euclidean norm weighted by positive constants $a_1, \ldots , a_m$. Let $L^\infty_0(d_i)$ be the space of measurable and essentially bounded real functions defined on $(0, d_i)$, and $L^1_\infty(0, d_i)$ be the functions from $L^\infty_0(d_i)$ whose distributional derivatives are also in $L^\infty_0(d_i)$. $L_2_0 = \prod_{i=1}^n L_2_i$ with the norm $\|f\|_2^2 = \sum_{i=1}^n \|f_i\|_2^2$, where $f_i$ are the components of $f$. The basic space for our approach is $X = L_2^n \times R^n_+$. For $u \in X$ we shall write $u = (u^a, u^b)$ with $u^a \in L_2^n$ and $u^b \in R^n_+$. If $H$ is a Hilbert space with the inner product $\langle \cdot , \cdot \rangle_H$ and $\mathcal{P}(H)$ is the set of all subsets of $H$, let $A : \mathcal{D}(A) \subset H \to \mathcal{P}(H)$, be a multivalued operator. $A$ is said to be monotone if, for all $u, v \in \mathcal{D}(A)$ and all $Au, Av \in \mathcal{P}(H)$ we have $\langle Au - Av, u - v \rangle_H \geq 0$. The monotone operator $A$ is "maximal monotone" if $A$ regarded as a subset of $H \times H$ (i.e. as a graph) is not properly included in any other monotone subset of $H \times H$. Let $\varphi : H \to (-\infty, +\infty)$ be a convex and proper (i.e. $\varphi \neq +\infty$) function. We define the subdifferential of $\varphi$ as the operator $\partial \varphi : H \to \mathcal{P}(H)$ with $\partial \varphi(u) = \{v \in H : \varphi(u) - \varphi(v) \leq \langle u - v \rangle_H \}$ for all $v \in H$.

3. Hypotheses and Results

First of all, let us state the assumptions that will be made below. So, the parameters from equations (E.)+(B.C.) will be restricted by:

(A1) For every $k = 1, \ldots , n$ we have $r_k \in W^1, \infty(0, d_k)$, and there exists $r_0^k$ such that $r_k(x) \geq r_0^k > 0$ almost everywhere (a.e.) in $(0, d_k)$.

For every $k = 1, \ldots , n$ we have $c_k \in L^\infty(0, d_k)$ and there exists $c_k^0$ such that $c_k(x) \geq c_k^0 > 0$ a.e. in $(0, d_k)$. For every $k = 1, \ldots , n, s_k > 0$.

The dielectric losses $g_k$ will be supposed to satisfy:

(A2) For every $k = 1, \ldots , n$, the function $x \to g_k(x, p)$ is in $L_2, k(0, d_k)$ for any $p \in R$.

For every $k = 1, \ldots , n$, the function $p \to g_k(x, p)$ is continuous from $R \times R$ for almost all $x \in (0, d_k)$.

There exists $K_1 > 0$ such that for all $k = 1, \ldots , n$ for all $p_1, p_2 \in R$ and almost all $x \in (0, d_k)$ we have $g_k(x, p_1) - g_k(x, p_2) \mid (p_1 - p_2) \geq K_1 (p_1 - p_2)^2$.

The resistive multiport is described by $-j \in G(u)$ (see Fig. 4), where:

(A3) $G : \mathcal{D}(G) \subset R^{2n+m} \to \mathcal{P}(R^{2n+m})$ can be "split" in $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$, where $\mathcal{D}(G_{11}) \cap \mathcal{D}(G_{21}) \subset R^{2n}$, $\mathcal{D}(G_{12}) \cap \mathcal{D}(G_{22}) \subset R^m$, and $\mathcal{D}(G) = \mathcal{D}(G_{11}) \cap \mathcal{D}(G_{21}) \times \mathcal{D}(G_{12}) \cap \mathcal{D}(G_{22})$.

There exists $K_2 > 0$ such that for all $x = (x^a, x^b)$, $y = (y^a, y^b)$ with $x^a, y^a \in \mathcal{D}(G_{11}) \cap \mathcal{D}(G_{21})$, $x^b, y^b \in \mathcal{D}(G_{22}) \cap \mathcal{D}(G_{12})$ and all $Gx \in G(x), Gy \in G(y)$ it holds:

$\langle Gx - Gy, x - y \rangle_{R^{2n+m}} \geq K_2 \|x^b - y^b\|_R^2$.

There exists $\varphi : R^{2n+m} \to (-\infty, +\infty)$, proper, convex and lower semicontinuous, such that $G = \partial \varphi$.

Now we are ready to enounce the main results of this paper. Everywhere we shall suppose (A1), (A2) and (A3) are fulfilled.

Firstly, about the dynamic solution we are able to prove:

**Theorem 1.** For any $v_0 \in L_2^n \times \mathcal{D}(G_{12}) \cap \mathcal{D}(G_{22})$ and any $T > 0$, there exists $v \in [0, T] \to X$, a unique solution for (E.)+(B.C.)+(I.C.) in the following sense:

- $u^a(0) = v_0^a$ and $u^b(0) = v_0^b$
- $v$ is absolutely continuous on each compact subinterval of $(0, T)$
- $v$ satisfies (E.)+(B.C.) a.e. on $(0, T)$, where the time derivatives are in the norm of $X$ while the space derivatives are in distribution sense.

In the second place we can establish:

**Theorem 2.** There exists $\overline{v} \in X$ a unique solution for the stationary problem.

Finally, about the asymptotic behaviour we have

**Theorem 3.** For any $t > 0$ and $v_0 \in L_2^n \times \mathcal{D}(G_{12}) \cap \mathcal{D}(G_{22})$

$\|v(t) - \overline{v}\|_X \leq \|v_0 - \overline{v}\|_X \exp(-K_3 t)$

where $K_3 = \min\{K_1; K_2/\min_{i=1}^m s_i\}$.

4. Comments

The hypotheses made above on line parameters $r_k(x)$ and $c_k(x)$ consist essentially in their boundedness in respect to the $x$-coordinate. Their discontinuities are practically unavoidable and this is one reason for the generalization of our models solutions.
The monotony of leakage current with respect to voltage (see (A2)) is also in accord with engineering practice.

The hypothesis (A3) on the resistive multiport is argued by the example presented in Chapter 1. Equations (4) show the existence of an operator $G : \mathbb{R}^3 \to \mathcal{P}(\mathbb{R}^3)$ that may be written in a matrix form

$$
Gw = \begin{bmatrix}
-f(V - \cdot) & 0 & 0 \\
0 & 1/R_1 & -1/R_1 \\
0 & -1/R_1 & 1/R_1 + 1/R_2
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix},
$$

where $f$ has the form (2) or (3). The monotony of $f$ implies $(Gx - Gy, x - y)_{\mathbb{R}^3} \geq \frac{1}{R_2} (x_3 - y_3)^2$. As $f$ is maximal monotone in $\mathbb{R}$ it follows that [15], there exists $\varphi_1 : \mathbb{R} \to (-\infty, +\infty]$ interior semicontinuous and convex such that $f = \partial \varphi_1$. The matrix $G^* = \begin{bmatrix}
1/R_1 & \cdots & 1/R_2 \\
-1/R_1 & \cdots & 1/R_2
\end{bmatrix}$ being symmetric and positive we have, [15], $G^* = \partial \varphi_2$ where $\varphi_2 : \mathbb{R}^2 \to (-\infty, +\infty]$ is proper, interior semicontinuous and convex. It follows that $G$ is maximal monotone and that $G = \partial \varphi$, where $\varphi(x_1, x_2, x_3) = \varphi_1(x_1) + \varphi_2(x_2, x_3)$, i.e. $G$ satisfies the hypothesis (A3).

We may conclude that the model (E.)+(B.C.)+(I.C.) restricted by (A1), (A2), (A3) is adequate to describe a large class of mixed type networks with distributed and lumped parameters appearing in integrated circuit modelling.

**REFERENCES**


